From Majorana Fermions to Topological Order

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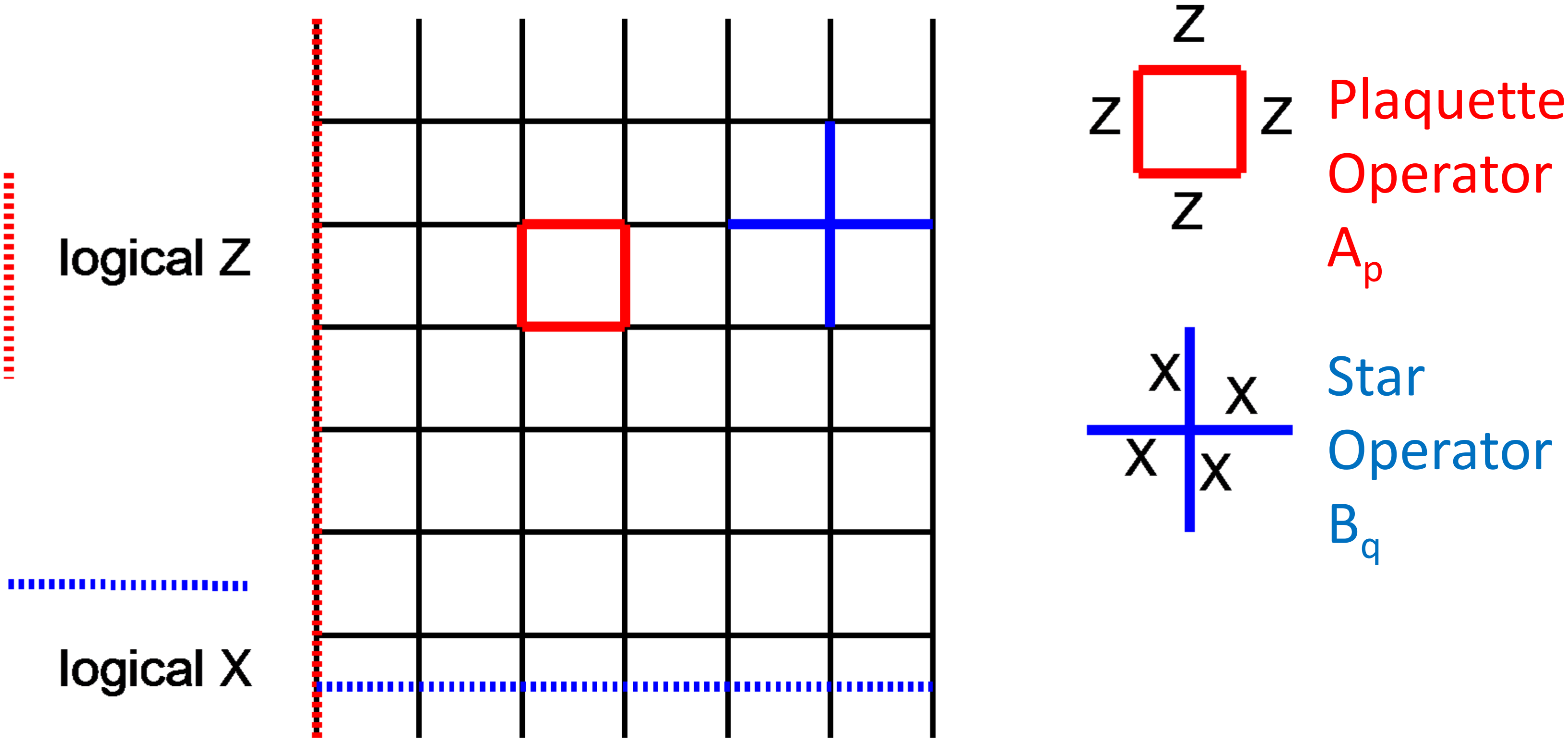
We are looking for PhD students or postdocs for theoretical research on coupling schemes & quantum error correction architectures for superconducting qubits.

Future of QC

- Any realistic effort towards a quantum computer will have to use some form of (topological) quantum error correction.
- Quantum error correction at its simplest is just choosing a protected representation of your qubit such that decoherence and relaxation are much (exponentially) suppressed.
- Favorite code architecture: surface code architecture (or variants thereof). Why?

Surface Code

n qubits on edges. Here n=85, L=7.



Example of a surface code which encodes 1 qubit into 85 qubits. One measures star and plaquette parity check operators and corrects if their eigenvalues are not all +1. Space where all eigenvalues are +1 is two-dimensional, a qubit , and logical operators X and Z on this encoded or logical qubit are non-local.

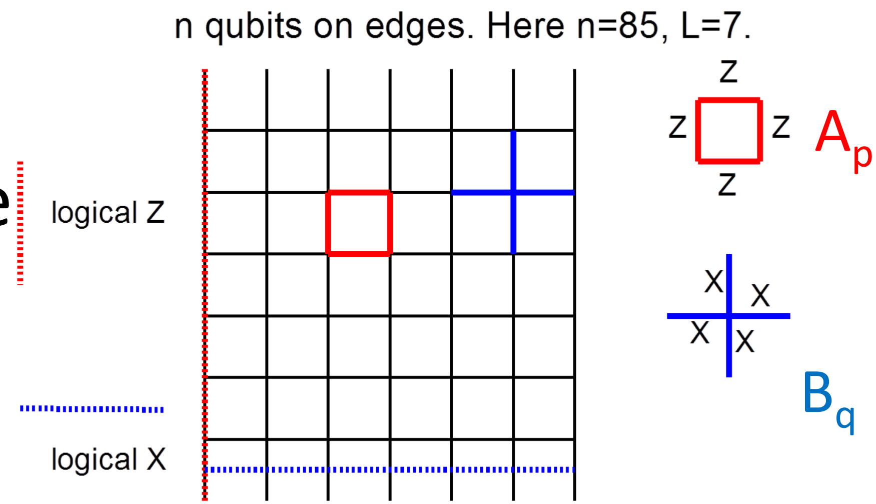
Surface Code Hamiltonian

- $H = -\sum_p A_p - \sum_q B_q$

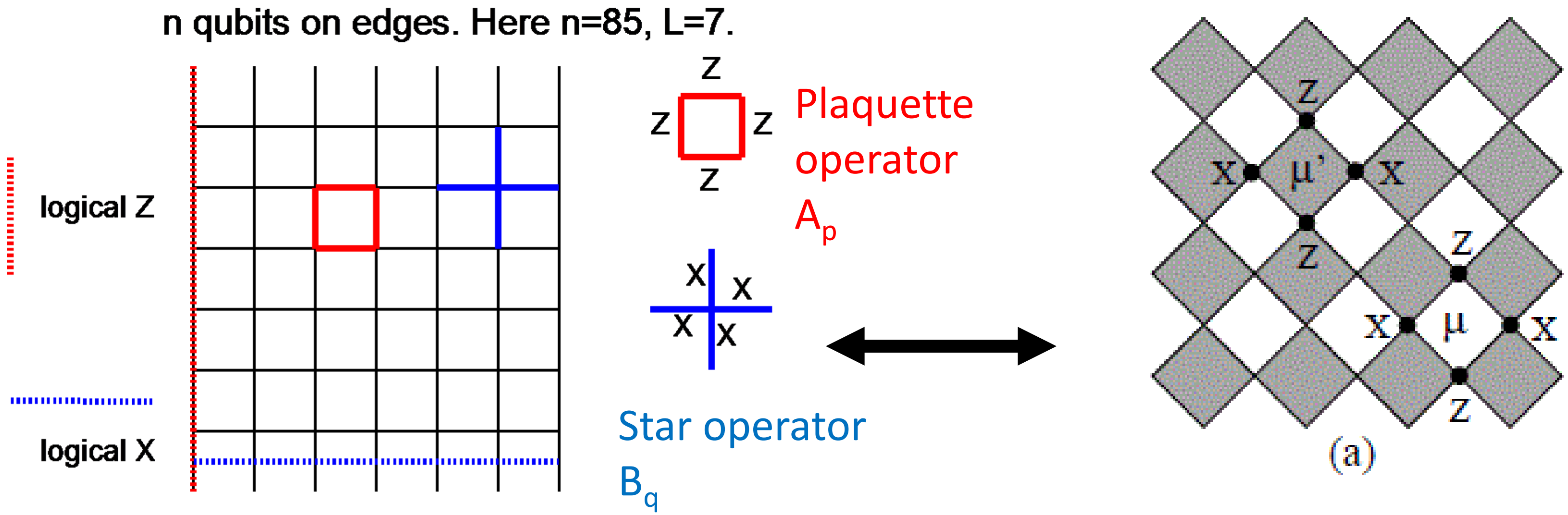
such that 2-dim ground-space is code space.

- H is gapped and

topologically-ordered: $H = -\sum_p A_p - \sum_q B_q + \varepsilon \sum_i V_i$ where V_i is any local term has energy-level splitting of ground-space $\sim \exp(-L)$ for $\varepsilon \leq \varepsilon_c$. Qubit is very protected against dephasing. Gap protects against thermal excitations, even better would be active driving towards ground-space via dissipative dynamics or error-correction.



Change of Basis



Local $X \leftrightarrow Z$ on vertical links gets you model on the right with qubits on vertices instead of links. Boundary conditions can be periodic (toric code encodes 2 topological qubits) or open (surface code, 1 qubit).

By making holes in the surface, i.e. omitting plaquette terms in the Hamiltonian H (or not measuring certain parity check operators), one can change degeneracy of the ground-space and encode multiple qubits.

Fowler, Stephens, Groszkowski, Phys. Rev. A 80, 052312 (2009)

Surface Code Architecture in a nutshell

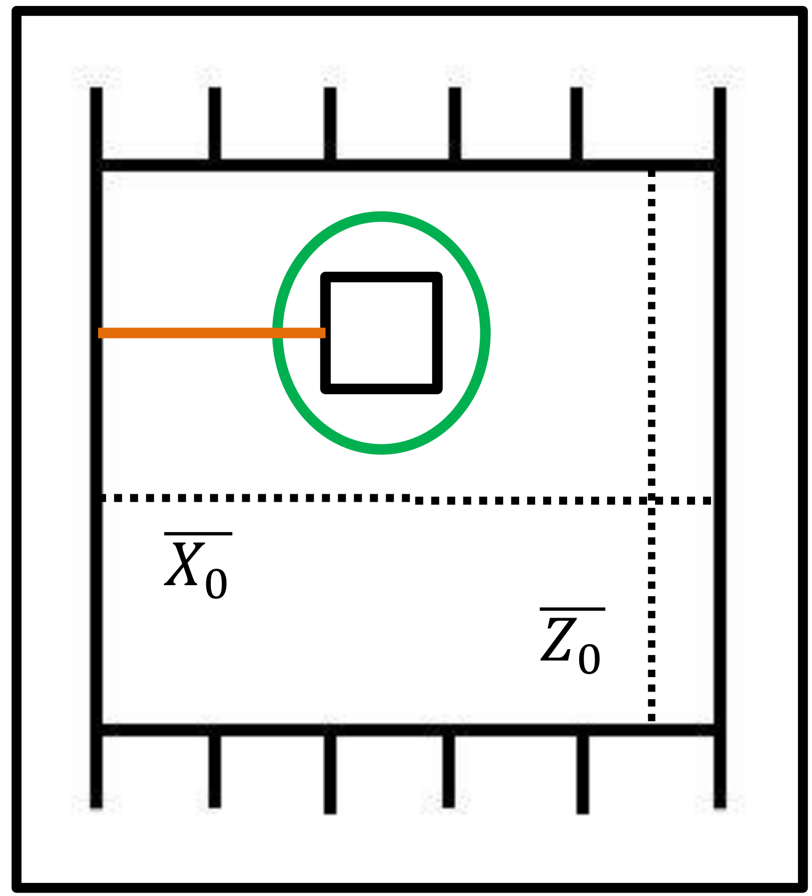
- Qubits layout on a 2D lattice. Each physical qubit is involved in 4 local parity check measurements, each involving 4 qubits. Each logical, encoded qubit takes up a patch O(L) x O(L) of the lattice.
- Architecture Level I: quantum memory⁺. Noise threshold is highest among known codes, goes to 0.75% for large lattice size L (e.g. $L=O(10^1)$). Hadamard and CNOT, are done topologically, just by changing which parity check operators to measure.
- Architecture Level II. Hadamard and CNOT are used to implement the S and T gate using the method of injection-and-distillation to get universality. This method has sizable overhead but the noise threshold for level II is $\sim$ 15%.
- Computation largely consists of parity check measurements, preparation and measurement of $|0\rangle, |1\rangle, |+\rangle, |-\rangle$ for qubits on the lattice.

How to encode more qubits

Make a bigger hole by taking out a volume of plaquette (and star operators) and their qubits. Hole has smooth boundary: smooth hole. Logical operators are green $\overline{Z_1}$ and orange $\overline{X_1}$.

Quality of qubit: minimum of circumference of hole and distance to smooth boundary of hole.

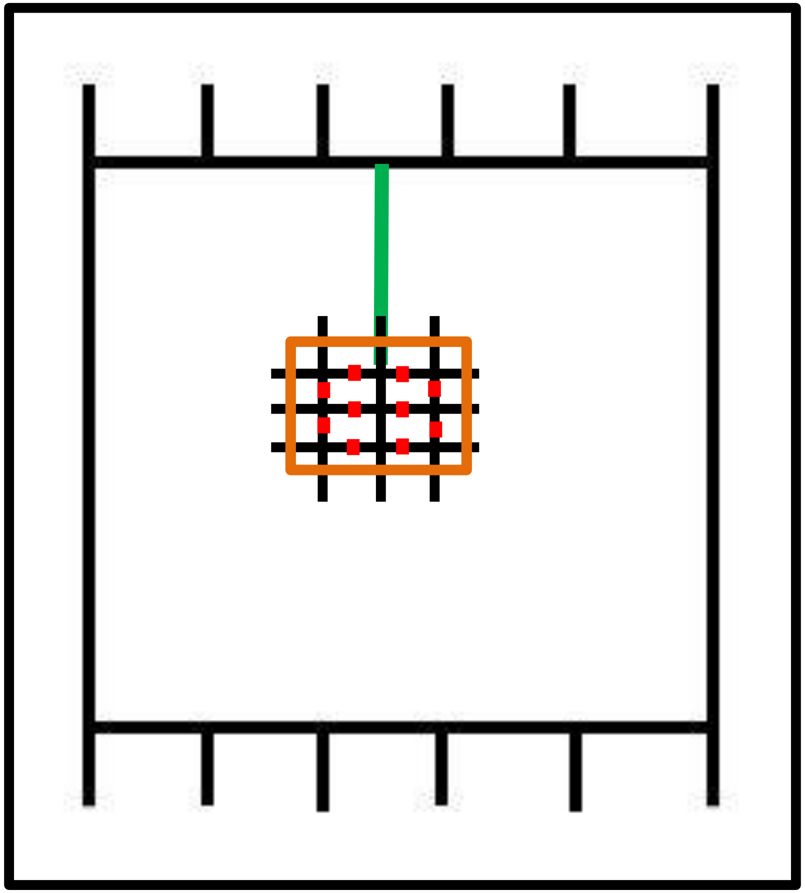
How to do a CNOT?



Rough hole

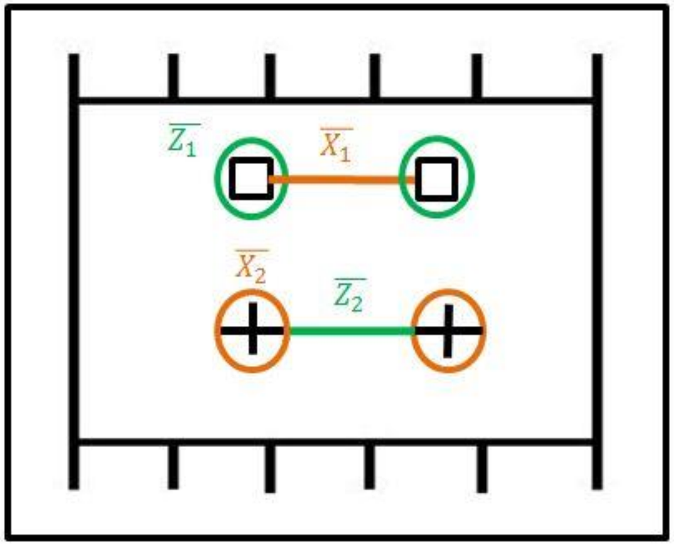
Similarly, one can make a rough hole by removing a cluster of star operators. It will have a rough boundary. E.g. in figure: we remove 9 star operators, 4 plaquette operators and 12 qubits.

Logical operators are green $\overline{Z_1}$ to rough boundary (rough-to-rough) and orange $\overline{X_1}$ which encircles the hole.



Coding Scheme

We make two smooth holes (2 qubits) of which we only use one: smooth qubit.
Similarly, we make two rough holes and use only one qubit.



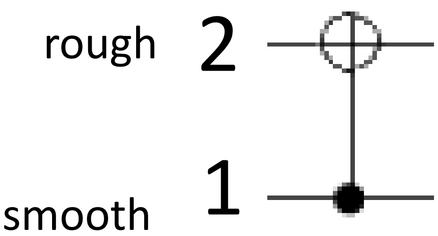
We encode the data qubits into the smooth qubits of sufficient large area and sufficiently far apart for protection. Rough qubits will be only used to help do CNOTs on smooth qubits.

Raussendorf, Harrington, PRL 98, 190504 (2007)

Raussendorf, Harrington, Goyal, New Journal of Physics 9, 199 (2007)

CNOT

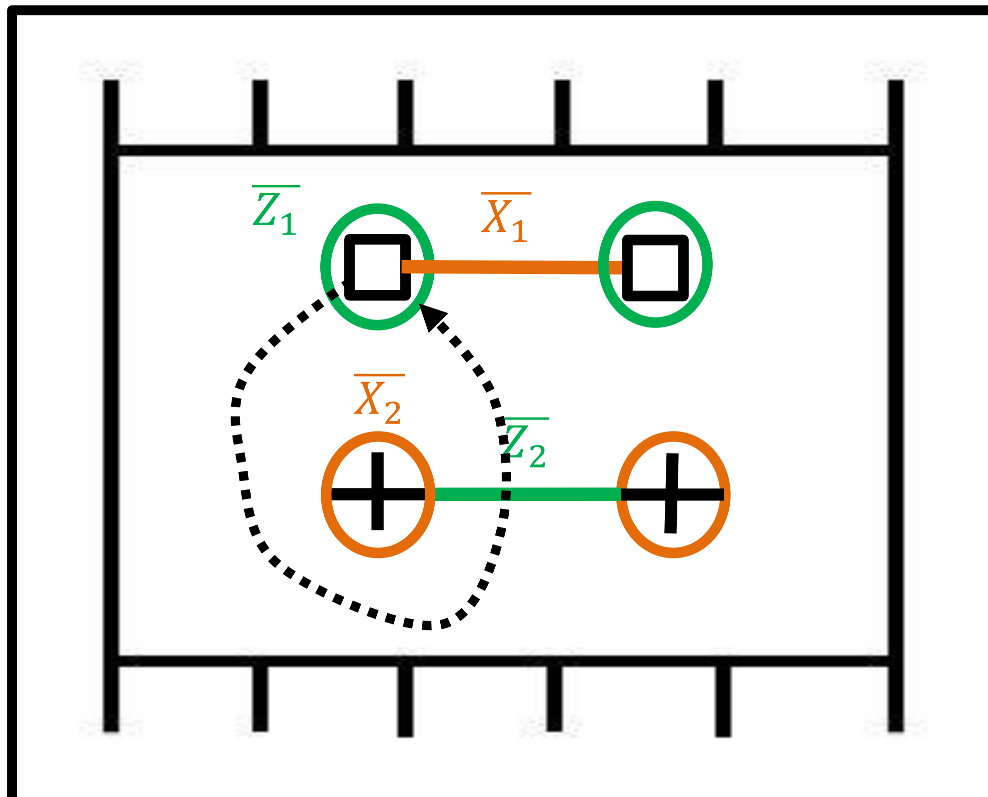
CNOT's action on Pauli's $UPU^{\dagger} = P'$



$$X_1 \otimes I_2 \rightarrow X_1 \otimes X_2$$
$$I_1 \otimes X_2 \rightarrow I_1 \otimes X_2$$
$$Z_1 \otimes I_2 \rightarrow Z_1 \otimes I_2$$
$$I_1 \otimes Z_2 \rightarrow Z_1 \otimes Z_2$$

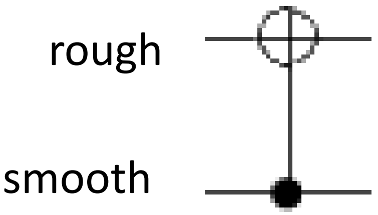
X-strings are orange.
Z-strings are green.
Control qubit is smooth qubit. Target qubit is rough qubit.

We move a smooth hole around a rough hole.
Deformation of Hamiltonian or code.



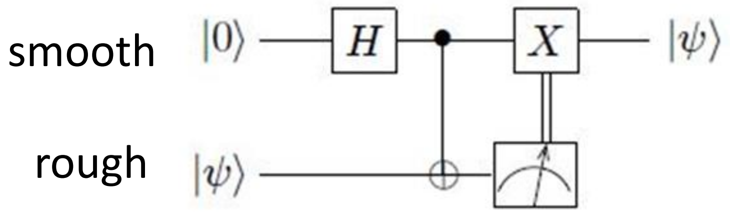
CNOT between smooth qubits

We can only do a CNOT between a smooth qubit as control and rough qubit as target qubit.
Looks quite limited....(such gates all commute!)

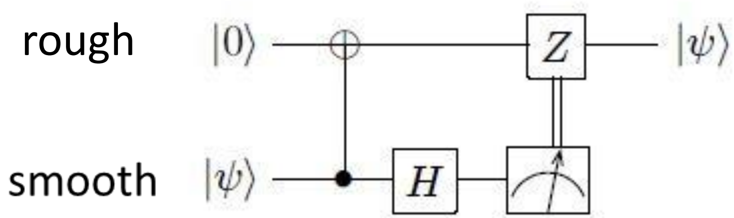


Needs a circuit trick.

1. Circuit for converting a rough qubit to a smooth qubit.



2. Circuit for converting a smooth qubit into a rough qubit.



How to do CNOT between (say) 2 smooth qubits:
Convert (target) smooth qubit into rough qubit using circuit 2, do CNOT and convert rough qubit to smooth qubit using circuit 1.

Comments

- Surface or toric code is a toy model where excitations/errors are Abelian anyons. Trick how to do CNOT is nontrivial and crucial.
- No need for non-Abelian anyons to get universality (extra overhead is drawback).
- More physical way: if encoding in surface or toric code is so useful, why don't we start with a physical system whose ground-space is the code-space instead of engineering all the couplings using basic 1- and 2-qubit gates. It is hard to come up with a system with 4-body interactions which is topologically-ordered!

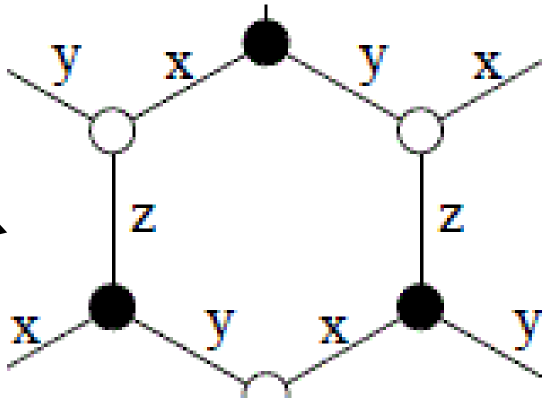
Work by Kitaev. Work by Ioffe, Doucot *et al.* on topologically-protected qubits in Josephson-junction arrays.

A. Kitaev, Annals of Physics 321 (2006) 2--111

Wishlist

- Physical system with effective Hamiltonian which is the toric or surface code,

e.g. Kitaev's honeycomb model in perturbative regime (vertical ZZ link much stronger than XX and YY links)



- Ability to make and move holes in the lattice by changing Hamiltonian H adiabatically (holes=absence of plaquette/star terms in H).
- Ability to prepare and measure single qubits on lattice in +, - and 0, 1 states.
- Ability to measure plaquette, star operators?

Interacting electrons

- Take Hubbard model $H = -t\sum_{<i,j>,\sigma}(c_{i,\sigma}{}^{\dagger}c_{j,\sigma} + h.c.) + U\sum_i n_{i,\uparrow}n_{i,\downarrow}$.

Perturbing around large U gives 2-body Heisenberg model… spin is in the way. Spinless fermions…

- Strong quartic interaction between two fermionic modes per site forcing electron parity to be odd, single electron occupying one of the two modes, |01> or |10>

$$H_0 = \Delta\left(-I + 2a_1{}^{\dagger}a_1\right)\left(-I + 2a_2{}^{\dagger}a_2\right)$$

- With $c_a = a_1 + a_1{}^{\dagger}, c_b = -i\left(a_1 - a_1{}^{\dagger}\right), c_c = a_2 + a_2{}^{\dagger}, c_d = -i\left(a_2 - a_2{}^{\dagger}\right)$ gives

$$H_0 = -\Delta\, c_a c_b c_c c_d.$$

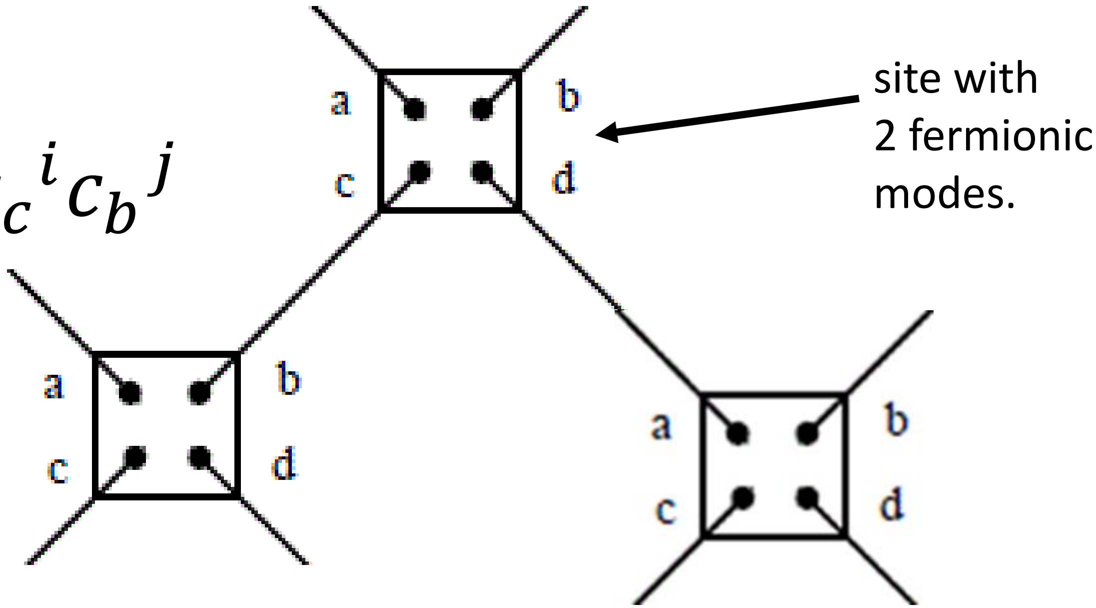
- One qubit per site.

Hopping?

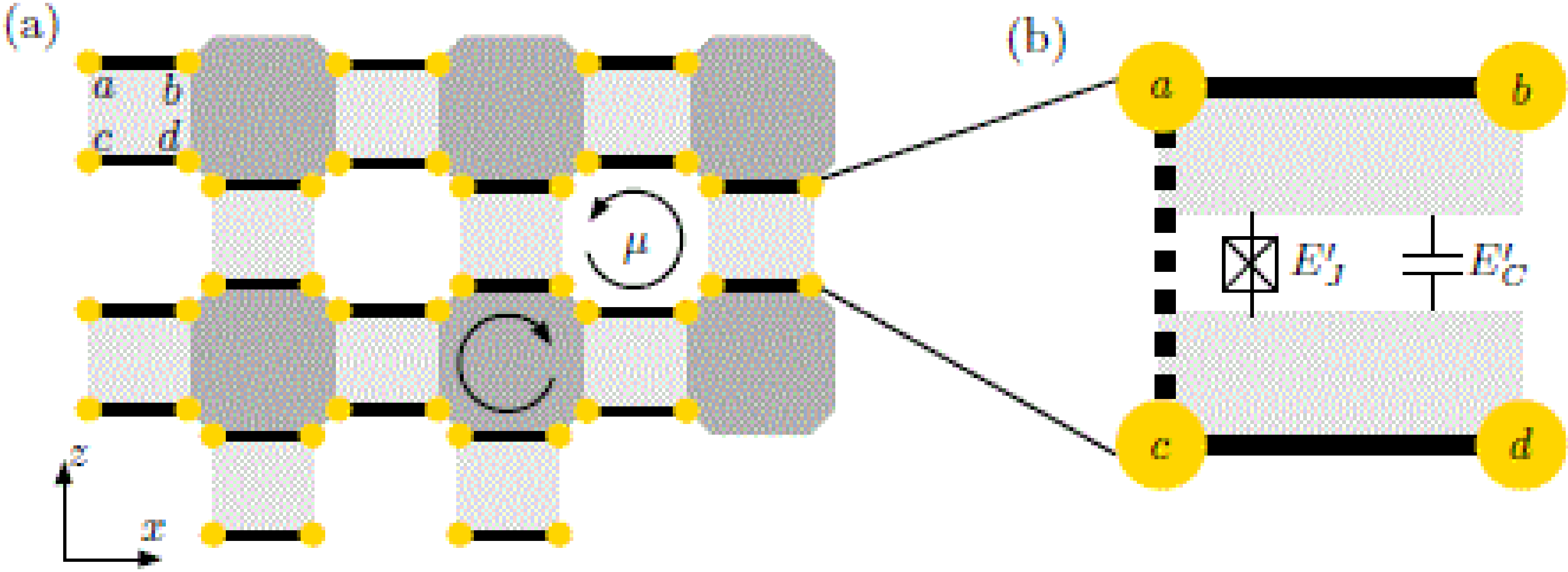
- Weak hopping term between sites i and j of the form $a_i a_j^\dagger + a_j a_i^\dagger$ would generate 2-body couplings between qubits on island i and j. Needed is superconductivity, i.e. $a_i^\dagger a_j^\dagger + a_j a_i$ so that couplings between sites are of the form $V = \pm\lambda\, i c_d{}^i c_a{}^j$

Similarly $V = \pm\lambda\, i\, c_c{}^i c_b{}^j$

etc.



Proposal for realization



Light-grey areas are superconducting (SC) islands, each with two InSb nanowires on top at the end of which are Majorana bound states: 4 Majorana fermions (in yellow), representing a single qubit. Tunneling coupling of strength λ for Majorana fermions between islands.

van Heck *et al*, New J. Phys. 14 (2012) 035019.

Physical mechanism for $H_0 = -\Delta\, c_a c_b c_c c_d$.

Each SC island capacitively coupled to a (common) ground-plate with $\mathrm{H} = E_C(n - n_{ind})^2 + E_J \cos\phi$ where n_{ind} is an offset-charge and ϕ is phase-difference. Regime $E_J \gg E_C$ where superconducting phase is fixed. Number of charges on island is a superposition of states with fixed parity. Favoring the charge on the SC to be odd by adjusting n_{ind} implies then favoring an odd number of electrons on wires, hence realizing H_0.

More precisely,

$\Delta \sim {E_C}^{1/4} {E_J}^{3/4} \cos(\pi\, n_{ind}) \exp(-\sqrt{8E_J/E_C})$.

A.Yu. Kitaev, Phys. Uspekhi 44, 131, 2001

Physical mechanism for tunneling

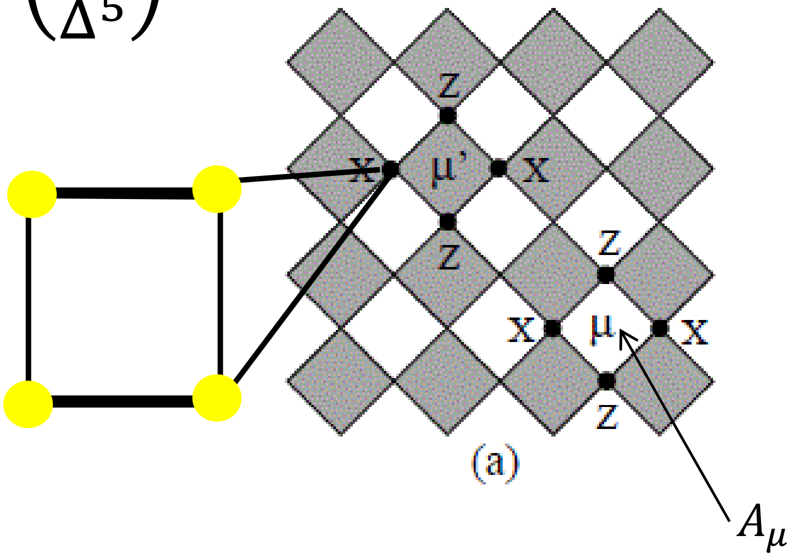
- Anomalous Josephson interaction through junction: $H_J = \lambda \sum_{i,j} V_{i,j} \cos((\phi_i - \phi_j)/2)$ with ϕ_i superconducting phase on island i, which we fixed by coupling all islands to the common ground superconductor. $V_{i,j} \sim i c_x{}^i c_y{}^j$, tunneling of single electrons in Majorana modes between SC islands.
- We assume that sign of $V_{i,j}$ is random, but fixed, not fluctuating.
- Tunneling coupling can be tuned by gate voltages, allowing to turn couplings on and off.

2D Interacting Majorana Fermion Model

- $H = H_0 + V$ with $H_0 = -\Delta \sum_i c_a{}^i c_b{}^i c_c{}^i c_d{}^i$ and $V = \lambda \sum_{i<j} V_{i,j}$ with $V_{i,j} \sim i\, c_x{}^i c_y{}^j$ and $\lambda \ll \Delta$.
- Perturbation theory analysis: first non-trivial terms due to tunneling occur in fourth-order and couple qubits on 4 islands:
- $H_{eff} = -\frac{5\lambda^4}{16\Delta^3} \sum_\mu A_\mu + O\left(\frac{\lambda^6}{\Delta^5}\right)$

$A_\mu = Z_{\mu+\hat{z}} X_{\mu+\hat{x}} Z_{\mu-\hat{z}} X_{\mu-\hat{x}}$

Toric (or surface) code Hamiltonian with 4-dim (or 2-dim) degenerate groundspace.



Advantage of Topological Qubit

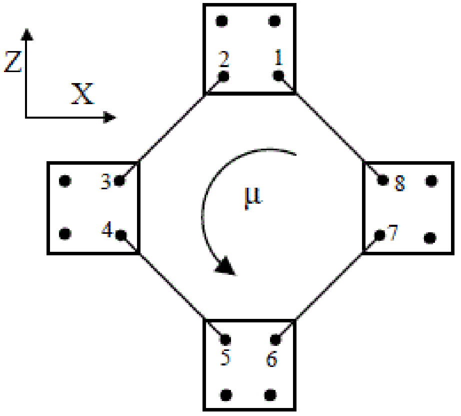
- One uses degenerate ground-space of $H = H_0 + V$ as qubit space. The degeneracy of this ground-space is insensitive to perturbations up to some critical strength as H is topologically-ordered. Perturbations: fluctuations in λ, Δ, quadratic terms coupling wire ends on the islands (strength ε). These lead to decoherence of Majorana fermion qubit in time ε^{-1}, while degeneracy of topological qubit is unaffected up to some critical ε_c.
- Not protected against quasi-particle poisoning which changes fermion-parity on island. But if this can be detected, one can convert this to regular error in the qubit space.
- CNOT can be done topologically.

Beyond Perturbation Theory

- Gap of effective toric code H_{eff} scales as $\frac{\lambda^4}{\Delta^3}$ so it is better not be in the perturbative regime when $\lambda \ll \Delta$.
- Will show that topological phase extends to critical point $\left(\frac{\lambda}{\Delta}\right)_c \approx 0.33$.
- Scales: Choosing $E_J \simeq 10K > \ E_C \simeq 5K$ so that $E_J \gg \Delta$ and $\lambda \leq 200\ mK$, and $\Delta \geq \lambda$. Gap $\backsimeq \lambda$ (current experiments at 50 mK).
- Hamiltonian has many conserved quantities, these can be used to map it onto a spin model.

Analysis

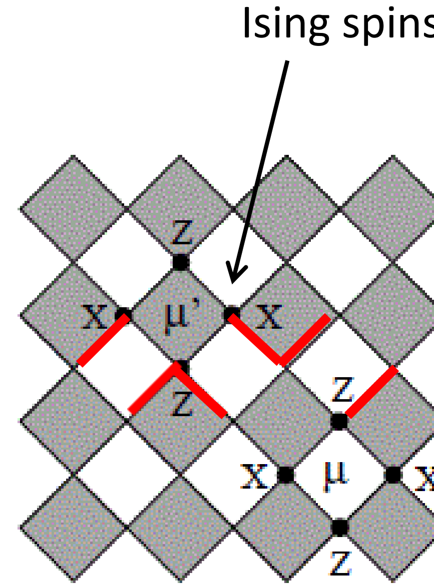
- Hamiltonian has many conserved quantities, these can be used to map it onto a 2-body spin model, e.g. $C_\mu = c_1 c_2 c_3 c_4 c_5 c_6 c_7 c_8$ for a plaquette μ commutes with H_0 and V. In qubit subspace, these act as the plaquette operators of the toric code.
- Rigorous analysis: Jordan-Wigner transformation to spin model leads to 2D transverse field Ising model with gauge bits capturing topological degrees of freedom.



Work by Fradkin, Srednicki, Susskind in 1980.

Phase Diagram

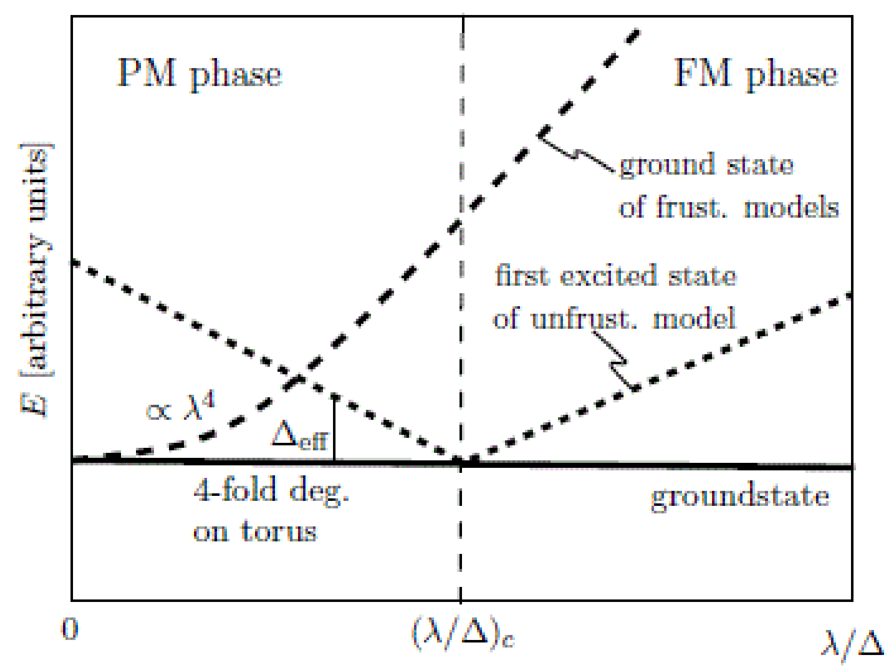
We map Majorana fermion model onto $H = -\lambda \sum_{<i,j>} \sigma_{i,j} Z_i Z_j - \Delta \sum_i X_i$ with $\sigma_{i,j} \in \{-1,1\}$. The $\sigma_{i,j}$ are gauge-bits which determine whether Ising spin interaction is ferro- or anti-ferromagnetic and whether there is frustration or not.



(a)

Frustration corresponds to the presence of anyonic excitations (frustrated parity checks/plaquettes)

Transition from topological to non-topological phase occurs at phase-transition of 2D transverse field Ising model: $\left(\frac{\lambda}{\Delta}\right)_c \approx 0.33$

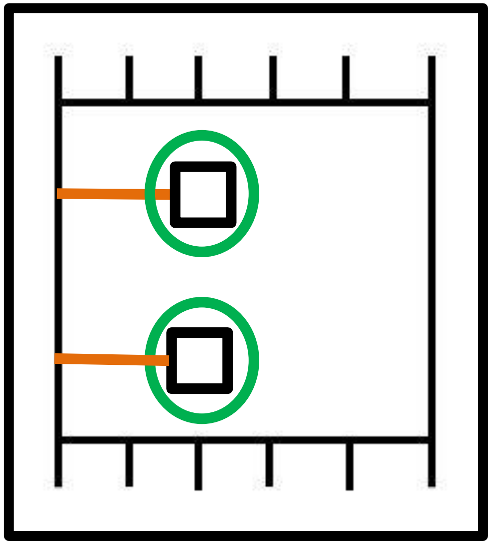


Conclusion

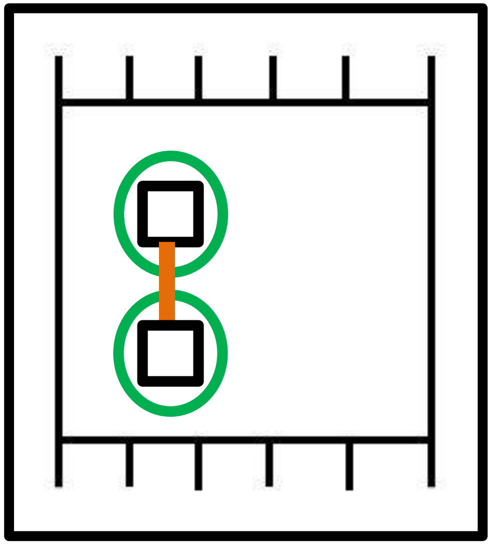
- We propose a way of using Majorana fermion qubits in 2D superconducting arrays so as to make topological qubits which realize the surface code architecture.
- Prep/Measure/Ancilla Prep of Majorana fermion qubits are used as building blocks in this architecture.
- Many open questions, e.g. can one measure plaquette operators C_μ for error correction?

More holes, new encoding

More holes, more qubits and we can deform their logical operators.
What is $\overline{X_1 X_2}$...
Deform to the orange logical operator $\overline{X_{smooth}}$ which connects the two holes.



No more reference to the boundary
(At least when 2 x distance to boundary is more than distance between holes).

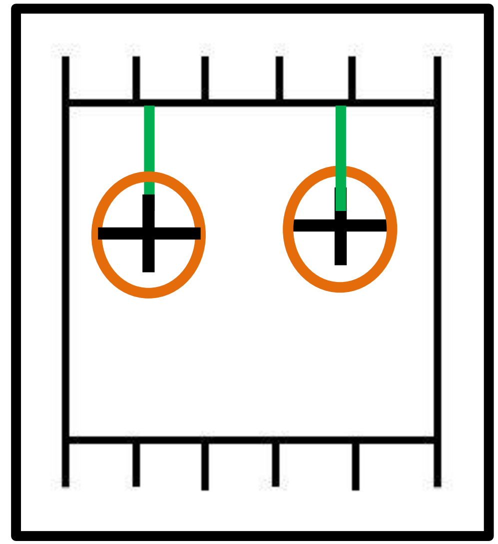


Smooth qubit is encoding of two hole qubits, i.e. $|\bar{0}\rangle \equiv |\bar{0}\rangle|\bar{0}\rangle$ and $|\bar{1}\rangle \equiv |\bar{1}\rangle|\bar{1}\rangle$.
$\overline{Z_{smooth}}$ is either $\overline{Z_1}$ or $\overline{Z_2}$.

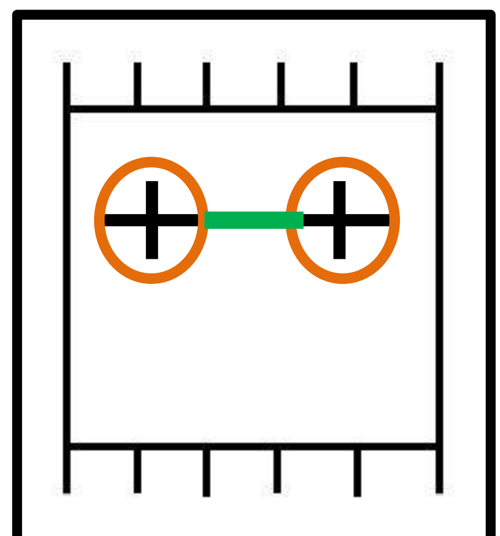
Many smooth qubits can be encoded.

Rough Qubits

Similarly, with two rough holes, one can encoded a rough qubit.
Deform green $\overline{Z_1 Z_2}$ to green $\overline{Z_{rough}}$ and $\overline{X_{rough}}$ is any of $\overline{X_1}$ or $\overline{X_2}$.



Thus the encoding of such a rough qubit is
$|\bar{0}\rangle \equiv |\bar{0}\rangle|\bar{0}\rangle + |\bar{1}\rangle|\bar{1}\rangle$ and
$|\bar{1}\rangle \equiv |\bar{0}\rangle|\bar{1}\rangle + |\bar{1}\rangle|\bar{0}\rangle$.



Creating and moving holes = changing the code = changing which parity checks we measure.