# Fermions and Quantum Computation: Some Theoretical Considerations

Barbara M. Terhal
IQI, Caltech &
IBM Watson Research Center

## References:


- B. Terhal & D. DiVincenzo, Classical simulation of noninteracting-fermion quantum circuits, PRA 65, 032325 (2002) http://xxx.lanl.gov/abs/quant-ph/0108010
- L. Valiant, Quantum computers that can be simulated classically in polynomial time, STOC '01.

# Motivation:

(1) Power of Quantum Computers

To understand the power of quantum computers, it is useful to understand when, -under what restrictions on the quantum circuit-, a quantum computer can be simulated efficiently by a classical device or is likely to be weaker than universal QC.

- Knill-Gottesman theorem: Clifford group operations, discrete subgroup of $U(2^n)$, can be simulated classically.
- Valiant's class of quantum circuits: Fermionic linear optics + measurements. (continuous subgroup of $U(2^n)$) can be simulated classically.
- Power of constant depth quantum circuits ? (this talk)

(2) What physical systems are capable of performing universal quantum computation?

Knill, Laflamme and Milburn (Nature **409**, 46, 2001).

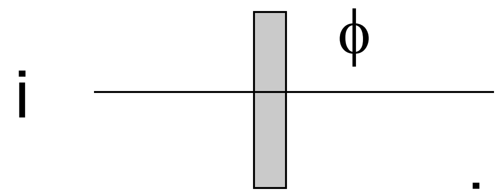
# Linear Optics with Fock states

1. Have single photon sources.
2. Can do photon counting measurement, detecting 0, 1 or 2+ photons.

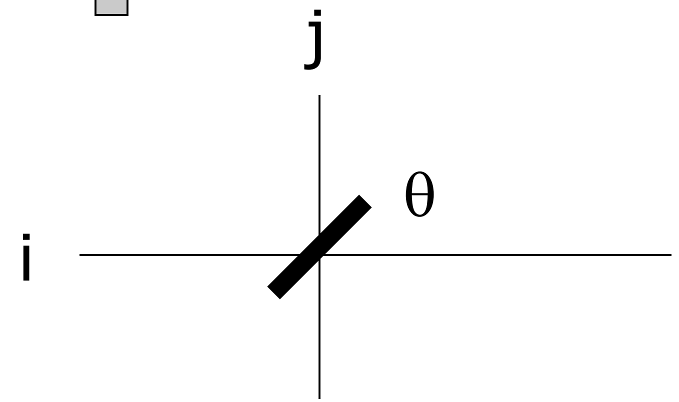
3. Circuit with passive linear optics elements: beamsplitters and phaseshifters, choice of gates can depend on previous measurement outcomes in circuit (adaptive).

1, 2 and 3 are sufficient to implement a quantum computer.

$$H^{i}_{phase} = a_i a_i^{+} \quad \textit{phaseshifter}$$




$$H^{i,j}_{beam} = a_i a_j^{+} + a_j a_i^{+} \quad \textit{beamsplitter}$$




$$\text{Encoding}: \quad |\bar{0}\rangle = |0\rangle_1 \otimes |1\rangle_2$$

$$|\bar{1}\rangle = |1\rangle_1 \otimes |0\rangle_2$$

# Other examples: spin systems

- Heisenberg interaction (2-qubit interaction) is universal by itself (DiVincenzo et al., Nature 408 (2000))

$$H_{heis}^{i,j} = \sigma_X^i \otimes \sigma_X^j + \sigma_Y^i \otimes \sigma_Y^j + \sigma_Z^i \otimes \sigma_Z^j$$

- Also universal is the interaction $H_{XY}^{i,j} = \sigma_X^i \otimes \sigma_X^j + \sigma_Y^i \otimes \sigma_Y^j$

- Also universal is 1-D nearest neighbor interaction

$$H_{XY}^{i,i+1} = \sigma_X^i \otimes \sigma_X^{i+1} + \sigma_Y^i \otimes \sigma_Y^{i+1}$$

plus arbitrary 1-qubit gates (Imamoglū et al. PRL 83 (1999))



1-D nearest neighbor interaction $H_{XY}^{i,i+1}$ only?

# Fermionic Linear Optics I

$|0\rangle_k$ no fermion in mode k

$|1\rangle_k$ 1 fermion in mode k

Spinless fermions!

We allow $H^i_{phase} = a_i a_i^+$ (phaseshifter), $H^{i,j}_{beam} = a_i a_j^+ + a_j a_i^+$ (beamsplitter)

$etc.$: anything quadratic in $a_i$ and $a_j^+$ plus fermion counting measurements

These interactions include nearest neighbor XY model.

$$\sigma_X^i \otimes \sigma_X^{i+1} + \sigma_Y^i \otimes \sigma_Y^{i+1} = \frac{-1}{2}\left(a_i a_{i+1}^+ + a_{i+1} a_i^+\right)$$

View qubits as fermionic modes, but creation/annihilation operators are nonlocal operations due to fermionic anti-commutation relations.

$$a_i \,|\, x >= 0,$$

$$a_i \,|\, x >= (-1)^{\oplus_{k=1}^{i-1} x_k} \,|\, x_1 \ldots x_{i-1}, \bar{x}_i, x_{i+1} \ldots x_n >$$

# Fermionic Linear Optics II

Spinless fermions: electrons in a high magnetic field.
Fermionic linear optics implementable by passive electron optical elements (changing the electrical energy landscape for the electrons).

Yamamoto group in Stanford: Quantum Electron Optics
W. Oliver et al. Science 284 (1999).
Electron beam splitter, bunching and antibunching depending on (anti)symmetry of spin part of the total wavefunction.

# Result

Theorem [Valiant, Terhal/DiVincenzo]
Fermionic Linear Optics has no added power over classical computation. We can efficiently simulate such a quantum circuit on a classical machine.

- A lot of entanglement can be created in such a computation, but that is not relevant.
- Difference between bosons and fermions!
- What extra interaction do we need to get universal QC with these spinless fermions?

# Extra quartic gate

What extra gate do we need?
Interaction that is cubic in the creation/annihiliation operations is nonphysical (does not preserve fermion number (mod 2))
We also need a two-fermion interaction such as
(Kitaev & Bravyi, quant-ph/0003137)

$$H_{\text{int}} = a_j^{+} a_j a_i^{+} a_i$$

Advantages for implementation (and a concrete implementation proposal) are not clear…

# Simulation

What does it mean to (classically) simulate a quantum circuit?

Be able to sample from the probability distribution that would arise as the result of any allowed measurement (after a sequence of allowed gates).

Consider the first measurement (on, say, first qubit). Compute probabilities for outcome 0 and 1. Flip coin biased according to those probabilities and fix outcome, say 0. Consider second measurement and calculate the probability that the second measurement has outcome 0 or 1, given that the first measurement has outcome 0. etc.

(Thus we need to calculate joint probabilities)

# How it works: example

Efficient representation of dynamics for fermion number preserving operations where Hamiltonian is quadratic in fermion creation/annihilation operators :

$$U a_i U^{-1} = \sum_{k=1}^{n} V_{ik} a_k$$

This representation corresponds to the easy diagonalization of fermionic linear optics Hamiltonian, but this is NOT ENOUGH by itself (compare linear optics that can give full QC).

Say, we want to compute <y|U|x> where x,y are bit-strings and U is a circuit corresponding to a sequence of fermionic linear optics gates that preserve fermion number.

# Example continued

$|x\rangle = a^\dagger_{i1} a^\dagger_{i2} \ldots a^\dagger_{ik} |0\rangle$ etc.

Use U|0>=|0>, x and y same Hamming weight.

$$\langle y | U | x \rangle = \sum_{m_1, m_2 \ldots, m_k} V_{i_1 m_1} \ldots V_{i_k m_k}$$

$$\langle 0 | a_{j_k} \ldots a_{j_1} a^+_{m_1} \ldots a^+_{m_k} | 0 \rangle =$$

$$\sum_\pi sign(\pi) V_{i_1 \pi(j_1)} \ldots V_{i_k \pi(j_k)} = \det(V')$$

Compute determinant of an, at most, n x n matrix in time < $n^3$

# General Case

Simulate fermion counting (0 or 1) measurement on subset of qubits.

Evaluate the probability that some subset of the qubits is in a particular state, $y^*$=10…0, given a starting state x=0110..

$$p(y^* \mid x) = \left\langle x \middle| U^{-1} a_{j_1} a_{j_1}^{+} \ldots a_{j_k}^{+} a_{j_k} U \middle| x \right\rangle$$

Use Wick's theorem to evaluate vacuum expectation value of sequence of annihilation and creation operators. Probability is Pffafian of a matrix which relates to the determinant and can be efficiently calculated.

# Intermediate models

Domain of
quantum computation

Intermediate models:
cannot be simulated
classically, but do not
give full quantum
computation either

Heisenberg interaction

**Adaptive** Linear Optics
with single photons

Certain constant depth
quantum computations

**Nonadaptive** Linear Optics
with single photons

Linear Optics on
Gaussian states and
p,q measurements

Clifford group gates

Fermionic Linear Optics

Domain of
classical computation

# Examples of intermediate models (for all we know)

- Knill-Laflamme 1 qubit model:

NMR quantum computation at high temperature:
One has 1 effective clean qubit and n other qubits at T=
but entirely coherent operations and $\langle \sigma_z \rangle$ measurements.

- **Nonadaptive** linear optics

Theorem [Terhal]: If one can classically simulate a nonadaptive QC model with destructive measurements by calculating measurement probabilities then one can also simulate the adaptive model.

So if the adaptive model=QC, then the nonadaptive model will not be easy to simulate classically.

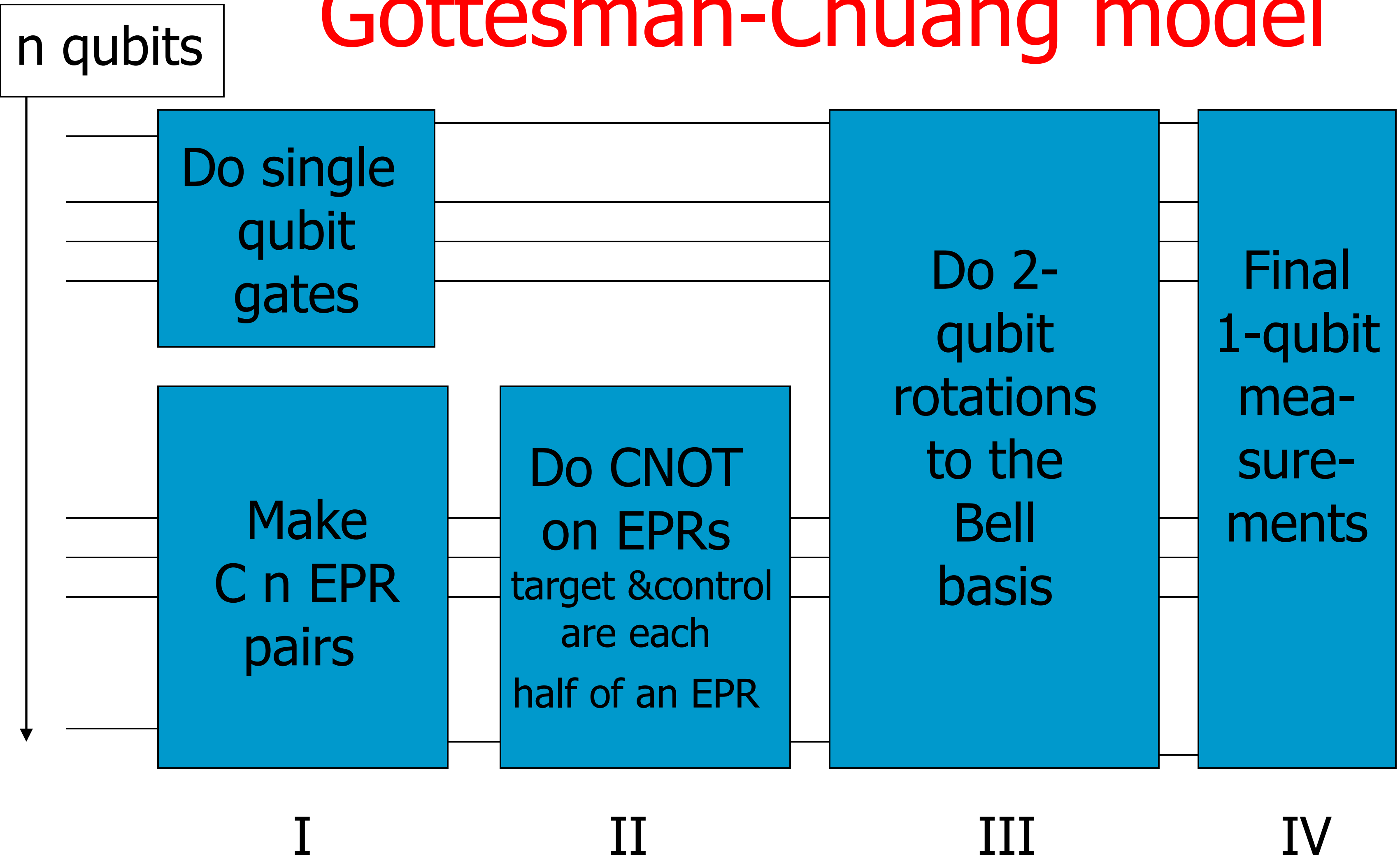
# **Constant depth** intermediate models

Other adaptive quantum computation models with destructive measurements:

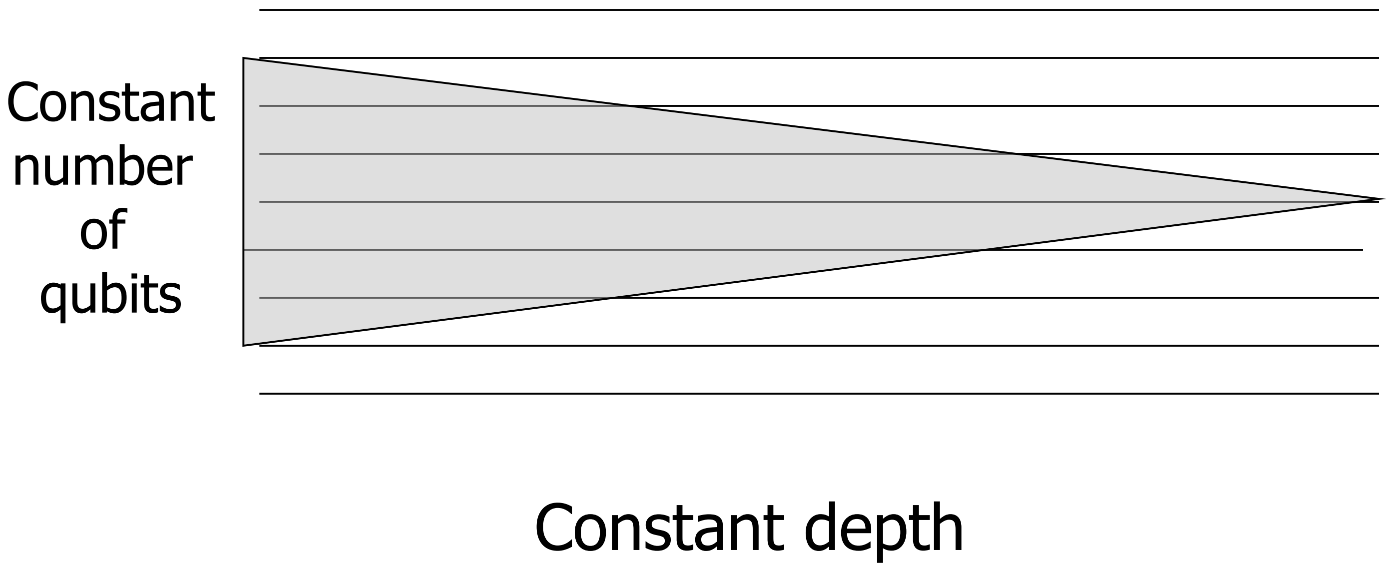
• Gottesman-Chuang (GC) model of quantum computation by teleportation. Every CNOT gate is applied offline on a set of EPR pairs and 'teleported into' the circuit by Bell measurements and single qubit Pauli-rotations.
• Raussendorf-Briegel (RB) model of computation of single qubit measurements on a (highly entangled) cluster state.

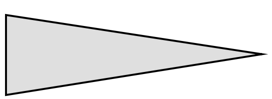
Nonadaptive versions of these models (move measurements to the end, no conditional operations) have circuit depth (using parallel processing) 4 (GC) and 8 (RB): in this number of steps you are done!

# Illustration for nonadaptive Gottesman-Chuang model



# Nontrivial correlations



Past history cone for qubit= all qubits that it has interacted with.

One can simulate log n qubits classically, but not the correlations in the outcomes of measurements on n qubits.

# Conclusion

- For fermionic implementation of quantum computation interaction that are quartic in fermion annihilation and creation operators are needed.
- There exist (as far as we know) constant (4 steps) depth quantum computations that do not have the full power of quantum computation but cannot be simulated classically (as far as we know). These are excellent candidates for first implementations. It will be interesting to see whether such constant depth computations can solve nontrivial problems or are of use in quantum protocols and tasks.