Universal conductance peak in the transport of electrons through a floating topological superconductor

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Contents

1 Introduction

In this work we study the transport of electrons through a junction between normal metal leads and a topological superconductor with a specific interest in the corresponding differential conductance. We will look in more detail at two different setups. First, we consider a grounded superconductor brought to tunneling connection to one normal lead (N-S junction) and then a floating superconductor connected to two normal leads (N-S-N junction). The superconductor is called floating because it is not directly connected to ground so that electrons only can enter or leave through the leads. In the following we will give a brief introduction to the three main topics this work is based on: the Landauer-Büttiker formalism, the BCS theory, and Majorana bound states in topological superconductors.

There are different approaches to describe the transport of electrons through a sample, e. g., the Kubo formula, Green's function method or the Landauer-Büttiker formalism (LB) [21]. We will use the latter to find the current through a junction between normal metal leads and a superconductor. The LB can be applied in case of coherent transport of noninteracting particles where coherent means that their motion across the system can be described by a wave function. Therefore, the size of the systems we consider is small enough to observe quantum mechanical effects, such as conductance quantization and contact resistance of an ideal lead and we shall return to this in section 3.1. A contradiction seams to arise from the restriction of the LB to noninteracting particles because the basic idea of superconductivity is that two electrons form a Cooper pair. Thus, the interaction between them can not be neglected. However, we will see that the mean field description of superconductivity provides a solution to this problem.

The general structure of the systems we consider in this work is a sample connected to particle reservoirs by normal, ideal leads. We apply voltages to the reservoirs so that their electrochemical potential, which includes the chemical and the electrical potential, may differ. This leads to a gradient and the electrons move across the sample to balance the differences. The LB is based on the ideas that the energy distribution of electrons emitted by a reservoir always follows the Fermi distribution of the corresponding reservoir and that electrons moving from the sample towards a reservoir enter it reflection-less [4]. Due to these assumptions we neglect the time dependence so that the problem of calculating the current becomes stationary and reduces to a scattering problem. We will solve the latter numerically.

If we compare the chemical and the electrical potential, we notice that the latter, since it interacts with every charged particle, changes the energy of each electron by an amount of $-eV$, where e is the elementary charge and V the applied voltage. Therefore, it shifts not just the Fermi energy, but also the band bottom. The chemical potential, on the contrary, depends on the particle density. If we increase it, it only will have the effect that the additional electrons occupy states above the original Fermi level thus the Fermi energy increases but the band bottom remains unaffected (cf. Fig. 1) [4]. Decreasing the density leads to the reversed process. In our case, we normally assume that the chemical potential is the same in all parts of the system so that the current arises only because of the applied voltage.

It should be noted that the LB neglects the influence of the electrical potential on the sample and thus on the scattering problem. If one does not apply this approximation, one will have to solve the Schrödinger equation and the Poisson equation self-consistently which is a ambitious problem [3]. However, we do not have to refer to it in this work because the considered voltages are very small.

The interpretation of current in the LB can be explained clearly by means of Fig. 1. The reservoirs emit electrons of each energy but weighted by the probability that the this state is occupied in the reservoir which is expressed by the corresponding Fermi distribution. The transmission coefficients that depend on the energy determine the fraction of transmitted and the reflected electrons that move across the sample. Finally, they enter reflection-less the reservoir to which they move. The net current is the current we are searching for. Normally, we are interested in the derivative of the current with respect to the applied voltage, called the differential conductance. In two or three-dimensional systems the wave function associated with the electron transport may have several traverse modes and the process described above takes

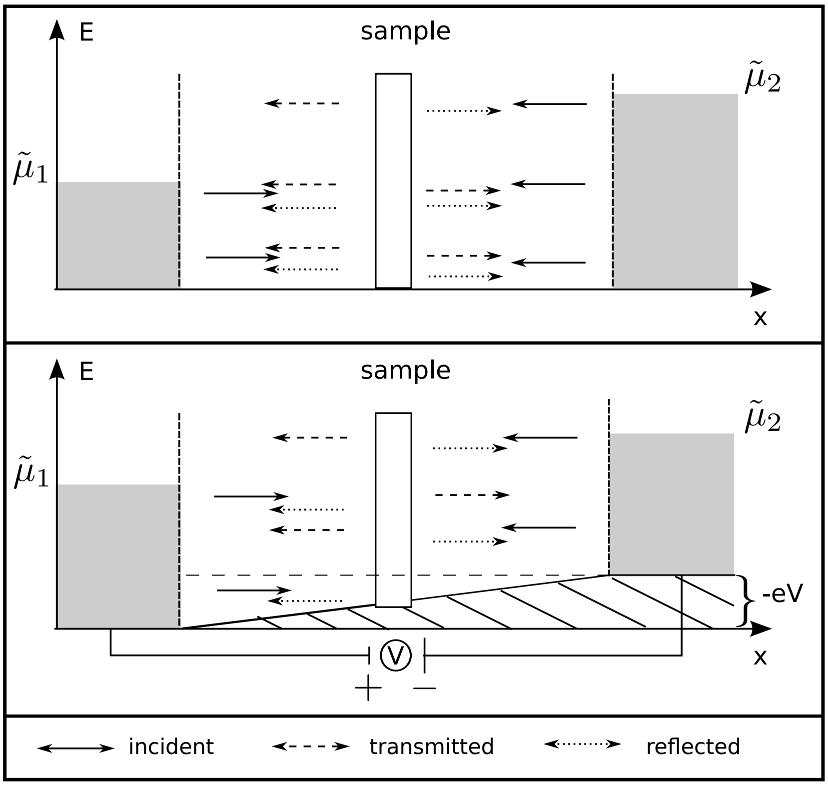

Figure 1: Sketch of the interpretation of the current in the Landauer-Büttiker formalism in case of two reservoirs held at chemical potentials $\tilde{\mu}_{1/2}$. At zero temperature, both reservoirs emit electrons (continuous arrows) with energies from the band bottom to the Fermi energy. Depending on the transmission coefficients, a fraction of the electrons at each energy is reflected (dotted) and transmitted (dashed). The electrochemical gradient arises from the difference of the chemical potentials (top) or the applied voltage V (bottom). Adapted from [21].

place independently for every mode at the same time [5]. However, we do not have to take this into account because we study the conductance of one-dimensional systems. Furthermore, we can transfer our results to higher dimensions as long as the voltage differences are to small to excite additional transversal modes.

The two crucial properties of a superconductor are the completely disappearing electrical resistance and the perfect diamagnetism which both can be phenomenologically described by the London equations. The microscopic approach is called BCS theory, named after J. Bardeen, L. N. Cooper, and R. Schriffer. The BCS theory is based on the assumption that pairs of electrons near the Fermi surface can form bound states due to an attractive potential so that the electrons can lower their energy by forming a so called Cooper pair. It follows that the excitation spectrum has a gap which separates the ground state from the remaining ones and leads to the electrodynamic properties. The most ideas that we summarize in this section are based on the presentation of the BCS theory in [19].

In the general case, we can write down the Hamiltonian of an electron gas with electron-electron interaction arising from the potential $V(\mathbf{r}_1 - \mathbf{r}_2)$ as

$$H = \sum_{\mathbf{k}\sigma} \xi_{\mathbf{k}} c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \frac{1}{2} \sum_{\mathbf{klq}\sigma} V(\mathbf{q})\, c^\dagger_{\mathbf{k+q},\sigma}\, c^\dagger_{\mathbf{l-q},\sigma}\, c_{\mathbf{l},\sigma}\, c_{\mathbf{k},\sigma}, \tag{1}$$

where $c^\dagger_{\mathbf{k}\sigma}$ is the creation operator of an electron with momentum $\mathbf{k}$ and spin σ, $\xi_{\mathbf{k}}$ are the single particle excitation energies, and $V(\mathbf{q})$ are the matrix elements of the potential associated with the momentum transfer q. However, it is not obvious how an attractive force between two electrons should arise because charges of equal sign are subject to the long-range Coulomb repulsion. The answer consist of two parts. On the one hand, an external electric potential is shielded in the presence of a free electron gas. The electrons will rearrange in such a way that the effect of the external potential in the electron gas decreases exponentially with the distance of its origin. This process is known as Thomas-Fermi screening and takes place on a length scale of the geometric mean of the Fermi wavelength and the Bohr radius [8]. Thus, if we describe the superconductor as consisting of a lattice of positive ions surrounded by an electron gas, the potential of the ions is screened as well as the repulsive Coulomb force between two electrons due to the other electrons . On the other hand the electrons can interact due to lattice vibrations whose quanta can be taken as quasiparticles, called phonons. Phonons carry a momentum and electrons are able to emit and to absorb them taking momentum conservation into account. Thus, electrons can interact due to momentum transfer by exchanging phonons and it turns out that the associated potential is attractive especially between electrons with zero total momentum and opposite spin (singlet pairing). Therefore, we can reduce Eq. (1) to the Hamiltonian

$$H_{\text{pair}} = \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \sum_{\mathbf{kl}} V_{\text{eff}}(\mathbf{k}-\mathbf{l}) c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow} c_{-\mathbf{l}\downarrow} c_{\mathbf{l}\uparrow}, \tag{2}$$

that neglects all scattering processes apart from these we just have mentioned. Because of the Thomas-Fermi screening, the attractive interaction dominates over the Coulomb repulsion and electrons form Cooper pairs.

However, H_{pair} is still to complex to obtain the spectrum. The number of electron pairs $(l\uparrow, -l\downarrow)$ that can occupy and leave again a given state $(k\uparrow, -\downarrow)$ is in order of 10^{19} in an ordinary superconductor [15] and H_{pair} describes all these interactions. Therefore one confines to the average occupation of the states and neglect the dynamics of the Cooper pairs which makes the problem manageable. This approximation is known as mean field description. The price is that the ground state does not consists of a fixed particle number and to fix at most its average one has to introduce a chemical potential of the superconductor [17]. From that point of view, the mean field Hamiltonian describes a superconducting reservoir. In context of electron transport this property means that the model of a superconductor has an intrinsic grounding because the particle number is always adapted to the chemical potential. This is the reason why we calculate the conductance for a grounded superconductor in section 3.2 by considering a N-S junction. To get rid of the grounding and to be able to tune the electrochemical potential independently of the ground, we have to attach a second normal lead on the other side and equate the currents through the leads. This approach enable us to calculate the conductance of a floating superconductor by making use of the advantage of the mean field description. We will treat this N-S-N junction in section 3.3.

To acquire a comprehension of topological superconductors and Majorana modes, it is useful to take a look on the excitation spectrum of a superconductor. From another point of view, the attractive potential leads to a rearrangement of the excitation spectrum [8]. In a simple model of a normal metal the dispersion relation is parabolic and in the ground state the single-particle states are occupied up to the Fermi energy. The spectrum is gapless and we can excite the system with infinitesimal energy cost. There are two distinguished single particle excitations we can imagine. The first is to occupy a state above the Fermi level which can be expressed as applying the corresponding creation operator. The second is to remove an electron from below the Fermi level by applying the annihilation operator. The generated hole can be taken as a new quasiparticle and the electron's annihilation operator as its creation operator.

For the excitation energies $E_{\mathbf{k}}$ of the superconductor one can find a relation with the excitations $\xi_{\mathbf{k}}$ of the normal metal which means that the total number of possible excitations is the same for the normal metal and the superconductor. We will have a closer look on the derivation in section 2.1 where we obtain

$$E_{\mathbf{k}} = \sqrt{\xi^2_{\mathbf{k}} + |\Delta|^2}. \tag{3}$$

We notice that even for $\xi_{\mathbf{k}} = 0$, it takes an energy of $E_{\mathbf{k}} = |\Delta|$ to excite the system. Hence, $|\Delta|$ is called the energy gap. The quasiparticles associated with this spectrum are named after

Bogoliubov and their creation and annihilation operator fulfill the Fermi commutation relation, hence, the Bogoliubov particle are Fermions. Moreover, they are are superpositions of electron and hole excitations [6]. For this reason, it is convenient to derive Eq. (3) from a Hamiltonian which describes the superconductor both in term of electrons and in terms of holes, providing particle hole symmetry. Due to this "doubled" description we not only obtain the excitation spectrum of the Bogoliubov particles but also of their antiparticles with a minus sign in Eq. (3) and one can take the spectrum as being symmetrical around the ground state and the excitations can be seen as a mixture of electron and hole excitations.

Majorana Fermions are named after Ettore Majorana who proposed a solution of a modified Dirac equation called Majorana equation. The Dirac equation describes Dirac Fermions which are massive particles with spin 1/2, e. g., electrons [25]. The solution of the original equation is a complex field. This property was taken as being very useful because the description of charged particles requires a complex field [29]. Moreover, it led to the prediction of antiparticles [26], e. g., positrons, with opposite charge and same mass that are the charge conjugated, in a way the complex conjugated, solutions of the equation.mIn contrast, the solution of the Majorana equation leads to a real field and therefore, it is not affected by complex conjugation so that it describes particles that are their own antiparticles, called Majorana fermions [29]. It is unclear whether there exist elementary particles with this property even though one can suppose that neutrinos are Majorana fermions [23].

In this work we refer to Majorana bound states which are zero energy excitations of a superconductor. A superconductor is a a natural place to search for Majorana zero modes because the excited states are a mixture of electron and hole excitations [2]. As we have pointad out, the excitation spectrum of a superconductor is symmetrical around the ground state and each excitation can be equivalently described as the creation of a quasiparticle with positive energy or the annihilation of its antiparticle with negative energy. Since Majorana fermions are their own antiparticles, thus the operators of creation and annihilation are the same, zero energy is the only possible excitation energy for Majorna zero modes [29].

Another remarkable property of Majorana states is that one can imagine an ordinary Dirac Fermion state as a composition of two MFs just as a complex number consists of two real numbers [2]. We can write $c^\dagger = \gamma_l + i\gamma_r$, where $c^\dagger$ is the creation operator of an electron and $\gamma_{l/r}$ are the Mojorana operators. As long as the two states $\gamma_{l/r}$ are closely together, one can not distinguish them from an ordinary fermion state [20]. However, a particular class of superconductors contains spatially separated pairs of Majorana states, where each pair defines a twofold degenerated fermion level at zero energy. The reason why these systems are of great interest to recent research is that they are able to store quantum information non-locally which might be a useful for quantum computing because the stored information is protected against local perturbations [13]. Furthermore, Majorana fermions are subject to non-Abelian exchange statistics under specific conditions, cf. [1]. The exchange of two indistinguishable, non-Abelian particles can lead to a different quantum state whereas the same procedure just multiplies the many-particle wave function by a non-relevant phase factor for Abelian particles [7], e. g., electrons and bosons [27]. Thus, one can manipulate information by interchanging the Majorana fermions [1]. The protection is called *topological protection* and hence, superconductors containing Majorana zero modes are known as *topological superconductors* [13].

Since in the normal superconductor the Bogoliubov quasiparticles consist of an excitation of a hole and an electron of different spin due to the singlet pairing, we notice that it is not possible to find there a quasiparticle which is its own antiparticle. Therefore, we consider effectively spinless electrons , e. g. electrons whose spins are all pointing in the same direction which form Cooper pairs via triplet pairing [20]. Since this kind of superconductivity is very fragile, experimental setups typically use a normal superconductor which induces triplet pairing in a semiconductor or an insulator placed on it. A magnetic field ensures that the all the spins are pointing in the same direction to gain effectively spinless electrons [10]. We concentrate on a one-dimensional superconducting wire. It will turn out that two unpaired Majorana states can occur at each side of the wire if the superconductor is held in its topological phase in contrast to the trivial phase without Majorana zero modes. A possible way to detect whether they are present or not, is to measure the conductance of a tunneling junction between the superconductor and normal leads which means using tunneling spectroscopy. If we apply a small voltage so that the energy of the electron is within the gap energy, one can assume that they enter the superconductor much more likely in the presence of Majorana zero modes because these are the only excitation states in this range of energy [10]. However, there are sub gap energy excitations in case of two or three dimension even for absent Majorana zero modes thus their experimental detection is not that easy [16].

There are many other works dealing with the conductance of topological superconductors, e. g., K. Flensberg [9], L. Fu [11], and R. Hützen [14],which make use of other methods than LB so that it worth to take a closer look at this topic and to compare their results to ours.

2 Model Hamiltonians

In this chapter we introduce the BCS Hamiltonian, which is a mean field Hamiltonian describing a superconducting condensate of Cooper pairs and its excitations. We go on with a short overview of the physics of Majorana fermions and illustrate the occurrence of Majorana zero modes at the end of a one-dimensional superconductor, using the example of the Kitaev model. Finally, the effective Hamiltonian of the weak interaction between the Majorana fermions of opposite sites of the superconductor is presented.

2.1 BCS Hamiltonian

In this section we follow the approach presented in [8]. The idea of the BCS theory is that two electrons near the Fermi surface with opposite spin and momentum can form a Cooper pair because of an attractive potential V_{eff}, which arises from electron-phonon interactions. These pairs of fermions are bosons and can lead to a superconducting phase. We can write down the pairing Hamiltonian H_{pair} of this model as

$$H_{\text{pair}} = \underbrace{\sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma}}_{\text{kinetic energy term}} + \underbrace{\sum_{\mathbf{kl}} V_{\text{eff}}(\mathbf{k}-\mathbf{l}) c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow} c_{-\mathbf{l}\downarrow} c_{\mathbf{l}\uparrow}}_{\text{Cooper pairing term}}, \tag{4}$$

where $c^{\dagger}_{\mathbf{k}\sigma}$ and $c_{\mathbf{k}\sigma}$ are the creation and annihilation operators of an electron with momentum $\mathbf{k}$ and spin σ. Moreover $\epsilon_{\mathbf{k}}$ denotes the spin-independent single particle excitation energies and $V_{\text{eff}}(\mathbf{k}-\mathbf{l})$ the matrix element of the potential associated with the momentum transfer $\mathbf{q} = \mathbf{k} - \mathbf{l}$. To obtain the excitation spectrum of the model by diagonalizing the Hamiltonian, we apply a mean field approximation. To emphasize the meaning of the Cooper pairing term in Eq. (4), we could have expressed it using the operators $\hat{b}^{\dagger}_{\mathbf{k}} = c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow}$ and $\hat{b}_{\mathbf{k}} = c_{-\mathbf{l}\downarrow} c_{\mathbf{l}\uparrow}$ that create/annihilate a pair of electrons with opposite spin and momentum. If we assume that the fluctuations of the number of electrons affected by V_{eff} are small compared to its expectation value, we can write

$$\hat{b}_{\mathbf{k}} = \underbrace{\langle b_{\mathbf{k}} \rangle}_{=b_{\mathbf{k}}} + \underbrace{(\hat{b}_{\mathbf{k}} - \langle b_{\mathbf{k}} \rangle)}_{\ll b_{\mathbf{k}}}. \tag{5}$$

Inserting this into Eq. (4) and just taking into account terms which are at most linear in the fluctuations, we can approximate the pairing Hamiltonian as

$$H_{\text{BCS}} = \sum_{\mathbf{k}\sigma} (\epsilon_{\mathbf{k}} - \mu) c^{\dagger}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \sum_{\mathbf{kl}} V_{\text{eff}}(\mathbf{k}-\mathbf{l}) (c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow} b_{\mathbf{l}} + b^{*}_{\mathbf{k}} c_{-\mathbf{l}\downarrow} c_{\mathbf{l}\uparrow} - b^{*}_{\mathbf{k}} b_{\mathbf{l}}). \tag{6}$$

Due to the mean field approximation, this Hamiltonian does not commute with $\hat{N} = \sum_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma}$, which means that the ground state $|\psi_{\text{BCS}}\rangle$ does not preserve the total number of particles. Indeed the pairing term can create a pair of electrons from nothing or annihilate a pair without creating an other one. Relating to the system we are considering, we can take these processes as pairs of electrons entering or leaving the superconducting condensate. However, the dynamics of the electrons of the condensate is not described by Eq. (6) because of the mean field approximation. From that point of view, the BCS Hamiltonian describes a superconducting reservoir and we have to add the term $-\mu\hat{N}$ where the chemical potential μ can be chosen so that the expectation value of $\hat{N}$ equals a given number N or, the other way around, μ determines the average number of particles [4, 8].

In the superconducting ground state electrons of opposite spin and momentum near the Fermi surface condense to Cooper pairs which means that they form bound states in the attractive potential V_{eff} with lower energy than in the normal states and leave behind an energy gap in the excitation spectrum. Introducing the energy gap $\Delta_{\mathbf{k}} = -\sum_{\mathbf{l}} V_{\text{eff}}(\mathbf{k}-\mathbf{l}) \langle c_{-\mathbf{l}\downarrow} c_{\mathbf{l}\uparrow} \rangle$, the BCS Hamiltonian can be written as

$$H_{\text{BCS}} = \sum_{\mathbf{k}\sigma} \xi_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \sum_{\mathbf{k}} (-\Delta_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow} - \Delta^{*}_{\mathbf{k}} c_{\mathbf{k}\uparrow} c_{-\mathbf{k}\downarrow} + b^{*}_{\mathbf{k}} \Delta_{\mathbf{k}}), \tag{7}$$

with $\xi_{\mathbf{k}} = \epsilon_{\mathbf{k}} - \mu$. Now, we will consider the excitation spectrum of the superconductor but it is not obvious how to deduce to it from Eq. (7) because of the two terms containing only creation respectively only annihilations operators. However H_{BCS} can be written as

$$H'_{\text{BCS}} = \sum_{\mathbf{k}} \Bigg[\begin{pmatrix} c^\dagger_{\mathbf{k}\uparrow} & c_{-\mathbf{k}\downarrow} \end{pmatrix} \underbrace{\begin{pmatrix} \xi_{\mathbf{k}} & -\Delta_{\mathbf{k}} \\ -\Delta^*_{\mathbf{k}} & -\xi_{\mathbf{k}} \end{pmatrix}}_{H_{\mathbf{k}}} \begin{pmatrix} c_{\mathbf{k}\uparrow} \\ c^\dagger_{-\mathbf{k}\downarrow} \end{pmatrix} + \xi_{\mathbf{k}} + \Delta_{\mathbf{k}} b^*_{\mathbf{k}} \Bigg], \tag{8}$$

which is a quadratic form and known as the Bogoliubov-de Gennes form. At a first glance, it seems that we have doubled the degrees of freedoms, because $H_{\mathbf{k}}$ has two eigenvalues and thus H'_{BCS} has $2n$, whereas H_{BCS} has n. But these additional eigenvalues are not independent since the two eigenvalues $E_{\mathbf{k}1}$ and $E_{\mathbf{k}2}$ of $H_{\mathbf{k}}$ fulfill $E_{\mathbf{k}1} = -E_{\mathbf{k}2}$. To gain the excitation spectrum, one can diagonalize $H_{\mathbf{k}}$ via the Bogoliubov transformation. We define the two operators $\beta_{\mathbf{k}\downarrow}$ and $\beta_{\mathbf{k}\uparrow}$ as linear combinations of $c^\dagger_{-\mathbf{k}\downarrow}$ and $c_{\mathbf{k}\uparrow}$ as follows

$$\beta_{\mathbf{k}\downarrow} = u_{\mathbf{k}} c_{\mathbf{k}\uparrow} + v_{\mathbf{k}} c^\dagger_{-\mathbf{k}\downarrow}\,, \qquad \beta_{\mathbf{k}\uparrow} = u_{\mathbf{k}} c_{\mathbf{k}\uparrow} - v_{\mathbf{k}} c^\dagger_{-\mathbf{k}\downarrow}. \tag{9}$$

If we choose the complex numbers u and v, so that they fulfill $|u_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2 = 1$, $u_{\mathbf{k}} = u_{-\mathbf{k}}$, and $v_{\mathbf{k}} = v_{-\mathbf{k}}$, the operators β will satisfy

$$\begin{aligned} \{\beta_{\mathbf{k}\uparrow}\,, \beta^\dagger_{\mathbf{k}\uparrow}\} &= \{u_{\mathbf{k}} c_{\mathbf{k}\uparrow} - v_{\mathbf{k}} c^\dagger_{-\mathbf{k}\downarrow}\,, u^*_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} - v^*_{\mathbf{k}} c_{-\mathbf{k}\downarrow}\} = |u_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2 = 1 \\ \{\beta_{\mathbf{k}\uparrow}\,, \beta_{-\mathbf{k}\downarrow}\} &= \{u_{\mathbf{k}} c_{\mathbf{k}\uparrow} - v_{\mathbf{k}} c^\dagger_{-\mathbf{k}\downarrow}\,, u_{-\mathbf{k}} c_{-\mathbf{k}\uparrow} + v_{-\mathbf{k}} c^\dagger_{\mathbf{k}\downarrow}\} = u_{\mathbf{k}} v_{-\mathbf{k}} - v_{\mathbf{k}} u_{-\mathbf{k}} = 0, \end{aligned} \tag{10}$$

which means that they obey the Fermi commutation relations. Inserting (9) into Eq. (8), $H_{\mathbf{k}}$ becomes a diagonal matrix and we obtain

$$H_{\text{BCS}} = \sum_{\mathbf{k}} (\xi_{\mathbf{k}} - E_{\mathbf{k}} + \Delta_{\mathbf{k}} b^*_{\mathbf{k}}) + \sum_{\mathbf{k}\sigma} E_{\mathbf{k}} \beta^\dagger_{\mathbf{k}\sigma} \beta_{\mathbf{k}\sigma}. \tag{11}$$

Since $\beta^\dagger_{-\mathbf{k}\downarrow}$ and $\beta_{\mathbf{k}\uparrow}$ obey the Fermi commutation relation, we are able to interpret this expression as a free fermion gas of Bogoliubov quasiparticles with the excitation energies $E_{\mathbf{k}}$. The excitation spectrum is derived in, e. g. [8] and is given by

$$E_{\mathbf{k}} = \sqrt{\xi^2_{\mathbf{k}} + |\Delta|^2}. \tag{12}$$

Therefore the superconducting ground state is the vacuum state in quasiparticle occupation number representation and its excitations are electrons that leave the superconducting condensate and become normal conducting, which might be seen as breaking the Cooper pairs. The minimum excitation energy is Δ and known as the gap energy.

2.2 Majorana zero modes

Another excitation of a system one is able to imagine, are Majorana fermions that are their own antiparticles. Since the creation of an particle can be taken as the annihilation of its antiparticle, the Majorana operators $\gamma^\dagger$ and γ have to fulfill $\gamma^\dagger = \gamma$. Whereas in the conventional BCS case Bogoliubov quasiparticles are a linear combination of creation and annihilation operators of opposite spin and momentum, Majorana fermions consist of operators with equal spin and momentum. From that point of view the Majorana Fermions are a special case of Bogoliubov quasiparticles [20]. Because no matter how to choose the coefficients u and v in Eq. (9), these Bogliubov quasiparticles will not satisfy the condition $\beta^\dagger = \beta$ and we have to consider a model of spinless electrons. Therefore we introduce introduce the Kitaev model of an one-dimensional tight binding superconductor with spinless electrons. Moreover it shows that Majorana fermions are not just a mathematical trick, but indeed will have a physical meaning. It consists of a chain with L sites, whose distance is set to 1 and that can be empty or occupied by an electron with energy cost μ. Further the electrons are able to hop between two nearest neighbor sites with an energy cost w. In this section we follow the presentation given in [18]. The Hamiltonian of an one-dimensional tight binding superconductor is given by:

$$H_{\text{chain}} = \sum_{j=1}^{L-1} \big[\underbrace{-w(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j)}_{\text{hopping term}} + \underbrace{\Delta(c_j c_{j+1} + c_{j+1}^\dagger c_j^\dagger)}_{\text{pairing term}} \big] - \underbrace{\mu \sum_{j=1}^{L} c_j^\dagger c_j}_{\text{onsite term}}, \tag{13}$$

where we identify the hopping term, the onside term and the pairing term, which is responsible for the Cooper pairing. Though we introduced in Eq. (6) Δ as a complex number, we can set its phase to zero without loss of generality, because we are always free to set the global phase (U(1) symmetry). Obviously, Eq. (13) does not preserves the total particle number for $|\Delta| > 0$ because it is the tight binding model of the BCS Hamiltonian and therefore includes the same mean field approximation. One can now define the Majorana fermions γ_{2j+1} and γ_{2j}, that satisfy the condition $\gamma_j^\dagger = \gamma_j$ and obey the Fermi commutation relation

$$\gamma_{2j-1} = \frac{1}{\sqrt{2}}(c_j + c_j^\dagger) \qquad \gamma_{2j} = \frac{i}{\sqrt{2}}(c_j - c_j^\dagger). \tag{14}$$

By solving Eq. (14) for $c_j^\dagger$ and c_j, we obtain

$$c_j^\dagger = \frac{1}{\sqrt{2}}(\gamma_{2j-1} - i\gamma_{2j}) \qquad c_j = \frac{1}{\sqrt{2}}(\gamma_{2j-1} + i\gamma_{2j}). \tag{15}$$

Thus, we may interpret a Majorana fermion as half of an ordinary Dirac fermion and conclude that they normally will appear in pairs. As long as the two partners of a pair are located closely to each other, we are not able to distinguish them from an ordinary fermion [20]. But the Hamiltonian (13) provides a way to get two single Majorana fermions at the two ends of the chain. If we set $\mu = 0$ and $w = \Delta$, H_{chain} reduces to

$$H_{\text{chain}} = 2iw \sum_{j=1}^{L-1} \gamma_{2j}\, \gamma_{2j+1}. \tag{16}$$

On the one hand we notice that now two Majorana fermions of neighboring sites are paired together: the second Majorana of each site is bound to the first one of the next site (cf. Fig. 2). On the other hand, considering the sum limits, we see, that the two Majoranas at the ends of the chain γ_1 and γ_{2L} do not enter the Hamiltonian, which means that they remain unpaired. We can combine γ_{2j+1} and γ_{2j} to a new creation operator $\tilde{c}_j^\dagger = \frac{1}{\sqrt{2}}(\gamma_{2j} - i\gamma_{2j+1})$, so that the Hamiltonian can be written in a diagonal form

$$H_{\text{chain}} = 2w \sum_{j=1}^{L-1} (\tilde{c}_j^\dagger \tilde{c}_j - \frac{1}{2}). \tag{17}$$

That way we can combine γ_1 and γ_{2L} as well, to get the creation operator $\tilde{c}_0^\dagger = \frac{1}{\sqrt{2}}(\gamma_1 + i\gamma_{2L})$, that excites a mode with zero energy cost. That is the reason for the name Majorana zero mode. Further it means, that the ground state is twofold degenerated [20]. Since we combined two Majorana fermion to get the operators $\tilde{c}^\dagger$ and $\tilde{c}$, they are the creation and annihilation operators of an ordinary Dirac fermion and obey the Fermi commutation relation. Therefore, we can deduce form Eq. (17) that the excitation spectrum of H_{chain}, apart from the Majorana zero mode, will consist of only one excited state at energy $E = 2w$, if we assume $\mu = 0$ and $w = \Delta$. For the general case and under the assumption of periodic boundary conditions, the spectrum is given in [18] and reads

$$E_{\text{k}} = \Big[\underbrace{(-2w\cos k - \mu)^2}_{\xi_{\text{k}}^2} + 4|\Delta|^2 \sin^2 k \Big]^{\frac{1}{2}}, \; -\pi \leq k \leq \pi. \tag{18}$$

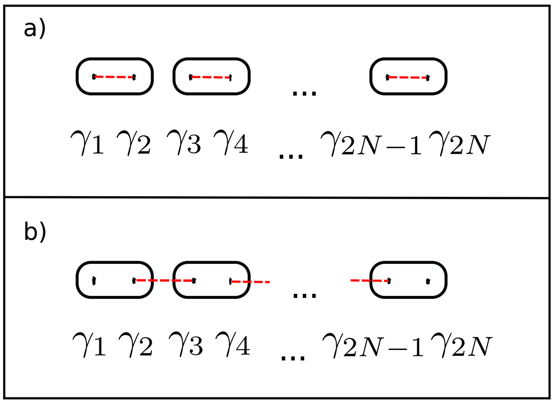

Figure 2: Sketch of the two phases of the Kitaev model. One box corresponds to one electron that can be taken as a composition of two Majorana fermions (dots). a) The trivial phase ($|\Delta| = w = 0$ and $\mu < 0$): The two Majorana fermions from one electron are paired together (dashed line) so that we can describe the system without assuming that the electron have an internal degree of freedom. b) The non-trivial phase ($|\Delta| = w > 0$ and $\mu = 0$): Two Majorana fermions of neighbored electrons are paired together, but the ones at the two ends of the superconductor remain unpaired, forming a Majorana zero mode. Adapted from [10].

ξ_k denotes the kinetic energy of a single, free particle with momentum k. Comparing this expression to the excitation spectrum of the BCS Hamiltonian in (12), we see that it has a similar form, but there is a crucial difference. If $2|w| = |\mu|$, the spectrum will not have an energy gap, no matter how we choose Δ. In case of $\Delta > 0$, the two regions $2|w| < |\mu|$ and $2|w| > |\mu|$ define two different phases of the superconductor and just the latter leads spatially separated Majorana states [18]. Henceforth the Majorana state at the left (right) end is referred to as γ_L (γ_R).

The above discussion about the occurrence of Majorana zero modes was done for the very specific case $\mu = 0$ and $w = \Delta$. For arbitrary values of μ, w, Δ, they are not exactly located at the two ends of the chain and there is an interaction between them that decreases exponentially with the chain length L. The Hamiltonian of this interaction is given by:

$$H_{\text{eff}} = \frac{i}{2} t\, \gamma_L \gamma_R, \qquad \text{where} \qquad t = \omega e^{-L/l_0}. \tag{19}$$

In this general case, γ_L and γ_R are a superposition of all the γ_i, wighted with coefficients, that are peaked at one end and decay exponentially towards the other one

$$\gamma_L = \sum_{i=1}^{2L} \alpha_{L,i}\, \gamma_i\,, \qquad \gamma_R = \sum_{i=1}^{2L} \alpha_{R,i}\, \gamma_i \tag{20}$$

(cf. Fig. 3). In this regard t in Eq. (19) can bee seen as a measure of the overlap of the two Majoranas zero modes.

At first glance the Hamiltonian (13) seems to describe a very artificial setup. But indeed, the qualitative properties of Majorana zero modes do not depend on the Kitaev model. It is sufficient to assume that it constitutes a rough approximation to which we add perturbations to describe a superconducting wire more precisely. We expect these perturbations to be local and that they can be expressed in terms of neighboring Majoranas $t_i \gamma_i \gamma_{i-1}$. But as long as $t \ll \Delta$, which means that we preserve the superconductivity, it will have no influence on the Majorana zero modes because the fermionic mode spanned by the two zero modes is highly non-localized with an exponentially weak interaction between both sides. The zero modes are said to be topologically protected.

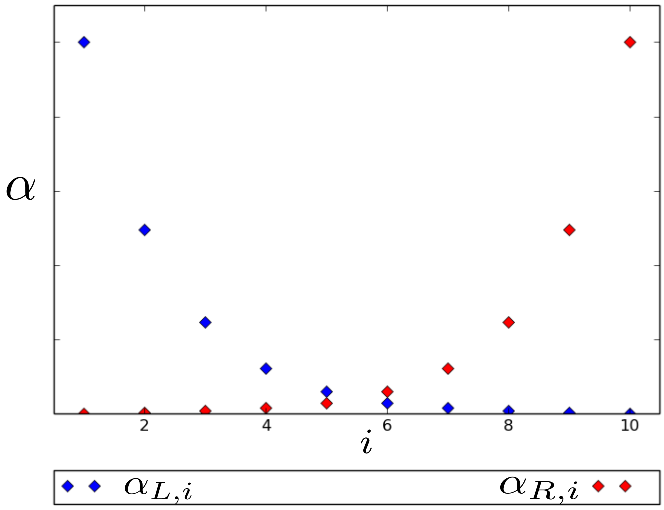

Figure 3: Sketch of the coefficients $\alpha_{L,i}$ and $\alpha_{R,i}$ for a chain with 5 sites. The coefficients are highly peaked around the first/last site.

3 Electron transport

In this chapter, we study the electron transport through a junction between a superconductor and one (N-S junction) or two (N-S-N junction) normal ideal leads. Our aim is to determine the differential conductance, that is the derivative of the current with respect to the applied voltage. We confine ourselves to small voltages $eV \ll \Delta$, where Δ denotes the gap energy. In addition we assume low temperatures $k_{\mathrm{B}}T \ll \Delta$ so that the superconductor can be described always by the BCS ground state.

To derive a formula for the current we introduce the basic results of the Landauer-Büttiker-formalism that reduces the problem of coherent electron transport to a scattering problem which can be solved numerically. At this point, we comment on the question why the current and the conductance of the junction between ideal leads do not reach infinite values.

The first system we consider is an N-S junction with a grounded superconductor that is in the topological phase. We compare the conductance peak in case of isolated and weakly coupled Majorana modes. Then we take a brief look at an grounded superconductor connected to two leads.

The second system we study in more detail is the N-S-N junction for the case of a floating superconductor. We derive a formula for the conductance measured with respect to the voltage difference between the two leads and for zero temperature. Finally, we prove that the conductance peaks at zero bias voltage, reaching the half of the maximal conduction of the N-S junction.

3.1 Laudauer-Büttiker-fomalism

At first sight, the calculation of a current through a lead seems to require a solution of a kinetic problem because the current is nothing else than charged particles that move between two reservoirs at different electrical potentials. But indeed, the Landauer-Büttiker formalism (LB) reduces the kinetic problem of coherent transport to a scattering problem of a single particle in a time independent potential. Coherent transport means that the transport can be described by a wave function. In the strict sense the LB formalism attributes the conductance of a sample that is connected to particle reservoirs by ideal leads to the transmission coefficients of a related transmission problem. Therefore, at first, we introduce the basic ideas of the scattering theory, following the presentation given in [24, 22], and then we describe the transmission problem which represents the particle transport in an arbitrary sample. Since we will restrict ourselves to only one conductance mode for the N-S as well as the N-S-N junction and assume that all the leads have the same chemical potential, we will cover only the single channel LB with symmetric leads.

Considering non-interacting spinless particles with mass m in a static real one-dimensional potential V that is nonzero only in a finite region, let's say for $|x| \le L$, we can write down the Schrödinger equation as

$$\Big[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+V(x)\Big]\,\psi(x)=E\,\psi(x)\;,\;\text{ with } V(x)=0,|x|\ge L \tag{21}$$

Since $V=0$ for $|x|\ge L$, we obtain superpositions of incoming and outgoing plain waves $\psi_\pm(x)$ for the asymptotic solutions of Eq. (21), $\psi_l(x)$ on the left side and $\psi_r(x)$ on the right side, far apart from the scattering region. To make sure that $\psi_\pm$ carries the unit probability flux, we choose the normalization

$$\psi_\pm(x)=\frac{1}{\sqrt{k}}\,e^{\pm ikx}\;. \tag{22}$$

Note that $\psi_\pm$ is also a function of energy E because $E=\frac{\hbar^2k^2}{2m}$. On the left side, $\psi_+(x)$ is an incoming and $\psi_-(x)$ an outgoing wave and on the right side, it is the other way around. Thus we obtain

$$\psi_l(x)=c_l^i\psi_+(x)+c_l^o\psi_-(x) \qquad \text{and} \qquad \psi_r(x)=c_r^i\psi_-(x)+c_r^o\psi_+(x), \tag{23}$$

where the upper index $i(o)$ stands for incoming (outgoing). To solve the Schrödinger equation (21), we have to compute the wave function taking into account the boundary conditions for ψ and its derivative. This leads us to a linear relation between the coefficients $c_l^i, c_l^o, c_r^i, c_r^o$, at which two of them can always be written as a function of the two remaining ones. One possible way is to relate the coefficients of the outgoing waves to the coefficients of the incoming waves. That defines the scattering matrix $S(E)$

$$\begin{pmatrix} c_l^o \\ c_r^o \end{pmatrix}=S(E)\begin{pmatrix} c_l^i \\ c_r^i \end{pmatrix}, \tag{24}$$

where $T_{ij}(E)=|S_{ij}(E)|^2$ is the transmission coefficient from lead j to lead i.
Now we consider a sample that is connected to N ideal leads that are connected to particle reservoirs, held at different chemical potentials μ_i. To calculate the current through a lead i, we take the sample as the scattering region with $|x|\le L$, cf. Eq. (21). The exact position of the boundary between the leads and the sample on the x-axis is not important because the leads just stand for the region of the asymptotic solution far from the scattering region where no scattering exists. Then we can compute the scattering matrix $S(E)$ and the transmission coefficients $T_{ij}(E)$. We give the formula for the current I_i through the i-th lead without a derivation. For particles with charge e and at finite temperature it is derived in [5] and given by

$$I_i=\frac{e}{h}\int dE\sum_{j=1}^{N}T_{ij}(E)\big[f_j(E)-f_i(E)\big], \qquad \text{where} \qquad f_j(E)=\frac{1}{1+e^{\beta(E-\mu_j)}} \tag{25}$$

is the Fermi-Dirac distribution of particles in the j-th reservoir, e the elementary charge, and $\beta=1/(k_BT)$. To gain a heuristic explication of Eq. (25), we follow the argumentation in [4] and write the current as

$$I_i=-\frac{e}{h}\int dE\,\big[f_i(E)-\sum_{j=1}^{N}T_{ij}(E)f_j(E)\big], \tag{26}$$

where we used $\sum_{j=1}^{N}T_{ij}(E)=1$, which is valid because of flux conservation. Assuming negatively charged particles, we find that every reservoir injects an incoming current of e/h per ΔE into its lead depending on the probability that the state at this energy is occupied in the reservoir and therefore wighted with the Fermi function. We have to subtract from that current the fraction that is reflected ($T_{ii}(E)$) and all the fractions that are transmitted from the other leads to lead i ($T_{i,j\neq i}(E)$) each wighted with the Fermi function of the appropriate reservoir. Usually, we will not create the potential difference between two leads by manipulating the chemical potential but by applying a voltage difference. In case of small voltage we can treat the electrical potential eV as an additional term and define the electrochemical potential $\tilde{\mu} = \mu + eV$ so that the current I_i in Eq. (25) and Eq. (26) is a function of V. In case of higher voltages the scattering potential will become a function of the voltage but for LB it is assumed to be static. In what follows, we normally assume that the chemical potential is the same in all parts of the system so that we can set it to zero because just the potential differences have a physical effect. Therefore, an applied voltage V can be written as $eV = \tilde{\mu_2} - \tilde{\mu_1}$. If we take into account the spin degeneracy of electrons, the current will double because electrons of different spin are not subject to the Pauli principle and they can carry charge independently. The transmission coefficients do not depend on the spin projection

It is remarkable that even if the sample is a normal ideal lead at zero temperature, the current will assume a finite value for a finite applied voltage $V = -\frac{1}{e}(\mu_2 - \mu_1) = V_2 - V_1$. If we consider this lead, connected to two other ones, we expect that there is no reflection ($T_{11}(E) = 0$) and a perfect transmission from lead 1 to 2 ($T_{12}(E) = 1$). Inserting this in Eq. (25) and assuming $T = 0$, we obtain

$$I_1(V) = \frac{e}{h} \int dE \left[\Theta(E - eV_2) - \Theta(E - eV_1) \right] = \frac{e^2}{h}(V_2 - V_1). \tag{27}$$

We can define the conductance $G(V) = \frac{\partial I_1}{\partial V}$ which leads us to the single-channel conductance in case of spinless particles

$$G(V) = \frac{e^2}{h}. \tag{28}$$

That is the half of the conductance quantum which is defined as $G_0 = 2e^2/h$ for the same system but with electrons carrying the current. This example seems to contradict the definition of an ideal lead that must not have any resistance and thus an infinite conductance. However, the definition of the conductance in Eq. (27) follows from the definition of where we apply the voltage difference. In the LB it is measured between the two particle reservoirs held at different electrochemical potentials. Hence, Eq. (27) is the conductance between the two reservoirs. For the case on an ideal lead it is impossible to create a voltage difference between two points of the wire. Therefore, to calculate the conductance of perfect wire with the LB, we have to consider an imperfect wire with a simple potential barrier so that $T_{11}(E) \neq 0$ and $T_{12}(E) \neq 1$. Indeed, the conductance will go to infinity if we set $T_{11}(E) = 0$ and $T_{12}(E) = 1$ afterwards, cf. [4]. From that we can conclude that the single channel conductance in Eq. (28) is the result of the junction between the leads and the reservoirs.

To apply Eq. (25) to a junction between a normal and a superconducting lead (N-S junction), we have to keep in mind that the current is carried by Cooper pairs in the superconductor for $eV \ll \Delta$ and by single electrons in the normal lead. In case of applied voltages that are smaller than the superconducting gap ($eV < \Delta$), a single electron can not enter the superconductor because there are no unoccupied single electron states for energies in the gap. But a pair of two electrons will be allowed to enter, if it forms a Cooper pair. To describe this process, we split each physical lead into an electron lead and a hole lead, cf. [6]. To gain a comprehension of this separation, we get back to the concept of particle hole symmetry we we already have mentioned in the introduction. Let us consider an electron reservoir at chemical potential μ. If we measure the electron energies with reference to the Fermi surface, we can take an excitation of an electron above the Fermi surface, and therefore with energy $\Delta E > 0$, as a hole with energy $-\Delta E$. From this point of view, the absence of an electron is a hole which carries the the opposite charge. Since the Fermi distribution indicates the probability of whether a state is occupied, we obtain

$$f_h(-E) = 1 - f_e(E) \tag{29}$$

for the relation between the Fermi distributions of the holes f_h and the electrons f_e. This separation between an electron and a hole lead allows us to describe the tunneling process into the superconductor at voltages smaller than the superconducting gap because the transmission of an electron as a hole, called Andreev reflection, can be taken as a pair of electrons entering the superconductor [6]. The reversed process is described by a hole reflected as an electron. Now we can use Eq. (25) to gain a formula for the current I_{Ne} through the normal electron lead. We measure all electrochemical potentials with reference to $\tilde{\mu_s}$ so that $\tilde{\mu_s} = 0$ and $f_{\mathrm{Sh}}(E) = f_{\mathrm{Se}}(E) = f_0(E)$. Thus we obtain

$$\begin{aligned} I_{\mathrm{Ne}} &= \frac{e}{h}\int_0^\infty dE\,\Big(T_{\mathrm{Ne,Nh}}(E)\big[f_{\mathrm{Nh}}(E) - f_{\mathrm{Ne}}(E)\big] + T_{\mathrm{Ne,Se}}(E)\big[f_0(E) - f_{\mathrm{Ne}}(E)\big] + \\ &\qquad\qquad + T_{\mathrm{Ne,Sh}}(E)\big[f_0(E) - f_{\mathrm{Ne}}(E)\big]\Big) \\ &= -\frac{e}{h}\int_0^\infty dE\,\Big(\big[1 - T_{\mathrm{Ne,Ne}}(E)\big]\big[f_{\mathrm{Ne}}(E) - f_0(E)\big] - T_{\mathrm{Ne,Hh}}(E)\big[f_{Nh}(E) - f_0(E)\big]\Big), \end{aligned} \tag{30}$$

where we used $T_{\mathrm{Ne,Ne}} + T_{\mathrm{Ne,Nh}} + T_{\mathrm{Ne,Se}} + T_{\mathrm{Ne,Sh}} = 1$, due to flux conservation, to write $T_{\mathrm{Ne,Se}} + T_{\mathrm{Ne,Sh}}$ in terms of the other two. In the same way the current in the hole lead I_{Nh} becomes

$$I_{\mathrm{Nh}} = \frac{e}{h}\int_0^\infty dE\,\Big(\big[1 - T_{\mathrm{Nh,Nh}}(E)\big]\big[f_{\mathrm{Nh}}(E) - f_0(E)\big] - T_{\mathrm{Nh,He}}(E)\big[f_{Ne}(E) - f_0(E)\big]\Big), \tag{31}$$

where we changed the sign of the current because of the positive charge of the holes. Notice that we evaluate the integral from zero to infinity, so that we do not count electrons and holes twice. If we set E to $-E$, we will find

$$I_{\mathrm{Nh}} = \frac{e}{h}\int_{-\infty}^0 dE\,\Big(\big[1 - T_{\mathrm{Nh,Nh}}(E)\big]\big[f_{\mathrm{Ne}}(E) - f_0(E)\big] - T_{\mathrm{Nh,Ne}}(E)\big[f_{Nh}(E) - f_0(E)\big]\Big), \tag{32}$$

by making use of Eq. (29) and taking into account that the transmission coefficients are even functions of the energy E. Because the system provides particle hole symmetry, the transmission coefficients fulfill $T_{\mathrm{Nh,Nh}} = T_{\mathrm{Ne,Ne}}$ and $T_{\mathrm{Ne,Nh}} = T_{\mathrm{Nh,Ne}}$. Hence we can write the current through the physical, normal lead I_N as

$$I_{\mathrm{N}} = I_{\mathrm{Ne}} + I_{\mathrm{Nh}} = -\frac{e}{h}\int_{-\infty}^\infty dE\Big(\big[1 - T_{\mathrm{Ne,Ne}}(E)\big]\big[f_{\mathrm{Ne}}(E) - f_0(E)\big] - T_{\mathrm{Nh,Ne}}(E)\big[f_{Nh}(E) - f_0(E)\big]\Big). \tag{33}$$

Finally we set E to $-E$ again in the second term and use (29), so that we obtain

$$I_{\mathrm{N}} = -\frac{1}{e}\int dE G(E)\big[f_{\mathrm{Ne}}(E) - f_0(E)\big], \qquad \text{where} \qquad G(E) = \frac{e^2}{h}\big[1 - T_{\mathrm{Ne,Ne}}(E) + T_{\mathrm{Nh,Ne}}(E)\big] \tag{34}$$

defines the zero temperature conductance of the N-S junction. This definition of the conductance makes sense because if we consider a the system at zero temperature and take μ of the normal lead as voltage V, we will find $\partial I(V)/\partial V = G(V)$. The other way around one can define the current as the integral of the conductance, weighted with Fermi functions.

3.2 Grounded superconductor

In this section we study the conductance of a grounded superconductor at zero temperature. We begin with a N-S junction, where the normal lead is in tunneling contact with a single isolated Majorana state at one end of the superconductor. Then we study the effect of a weak coupling of the two Majorana states at the opposite sides of the superconductor on the conductance. In both cases we compare the numerical results with the analytic formula by Flensberg. Finally we take a look at a grounded superconductor that is connected to two leads.

3.2.1 Isolated Majorana bound states

As we pointed out in section 2.1, a superconductor, described by the mean field BCS Hamiltonian (6) or by the tight binding Hamiltonian (13), can be seen as a reservoir held at an chemical potential μ_s. Therefore a simple N-S junction describes a normal lead connected to a grounded superconductor and we can calculate the conductance via Eq. (34). We use the *Python* library *Kwant* [12] to compute the transmission coefficients $T_{\mathrm{Ne,Ne}}$ and $T_{\mathrm{Ne,Nh}}$ numerically. We implement the scattering potential as a tight binding model that consists of three parts: the normal lead (N), the superconductor (S), and the tunneling contact (γ) between them. The corresponding Hamiltonian is given by

$$H_{\text{N-S}} = \underbrace{-w \sum_{j=-\infty}^{0} (c_{j+1}^\dagger c_i + h.c.)}_{H_{\mathrm{N}}} \underbrace{-\gamma(c_1^\dagger c_0 + h.c.) +}_{H_{\mathrm{t}}} \\ \underbrace{+ \sum_{j=1}^{L-1} \Big[-w\,(c_{j+1}^\dagger c_i + h.c.) - \mu_s c_i^\dagger c_i + \Delta(c_{i+1} c_i + h.c.) \Big]}_{H_{\mathrm{S}}} \tag{35}$$

which is the sum of the Hamiltonians of these three regions. We notice that L is the length, w the hopping parameter and μ_s the chemical potential of the superconductor. Moreover, Δ denotes the superconducting gap and γ the strength of the tunneling contact. We can take w as the intrinsic energy scale of the system and measure the other parameters in units of it. As we have pointed out in section 2.2, the choice of $\mu_s = 0$ and $w = |\Delta|$ leads to the occurrence of two single, isolated Majorana fermions at the ends of the superconductor.

In case of uncoupled Majorana state the analytic formula for the conductance $G(V)$ is is derived in [9] by Flensberg and is given by

$$G(V) = \frac{\partial I}{\partial V} = \frac{2e^2}{h} \frac{4\Gamma^2}{(eV)^2 + 4\Gamma^2}, \tag{36}$$

where $\Gamma/\hbar$ is the rate of transitions between the normal lead and the superconductor and is assumed to be independent of the applied voltage. Moreover, $\Gamma \ll \Delta$ so that the superconducting system always has enough time to return to its ground state. For the one-dimensional tight binding model Γ can be expressed in terms of the coupling strength γ and the density of states ρ_{N} of the normal lead by making use of Fermi's golden rule. We obtain

$$\Gamma = 2\pi|\gamma|^2 \rho_{\mathrm{N}}, \tag{37}$$

where we assume that the density of states in the normal lead is assumed to be constant for small bias voltages. To be able to compare the analytic formula with the numerical results, we have to express $\rho_{\mathrm{N}} = \partial n(E)/\partial E$, where $n(E)$ is the number of states per length at energy E, the transition rate in terms of the parameters of the tight binding model. n is given by $n = \frac{1}{L}\sum_k \Theta(E - E(k))$ which means that we count the momentums up to energy E. We can replace the sum by the integral $\frac{L}{2\pi}\int dk$ which leads us to

$$\rho_{\mathrm{N}} = \frac{\partial n}{\partial E} = \frac{1}{2\pi} \int dk\, \delta(E - E(k)). \tag{38}$$

The tight binding dispersion relation in the leads is given by

$$E(k) = \underbrace{-2w_{\mathrm{N}} \cos(k)}_{\epsilon_{\mathrm{kin}}} - \mu_{\mathrm{N}}, \tag{39}$$

which we can use to replace the integral over k by an integral over E. Taking into account that there are two values $\pm k$ for every value of E, we obtain

$$\rho_{\mathrm{N}}(E) = \frac{1}{2w \sin\Big(\arccos\big(-\frac{E+\mu_{\mathrm{N}}}{2w}\big)\Big)} = \frac{1}{\pi\sqrt{4w^2 - (E+\mu_{\mathrm{N}})^2}}. \tag{40}$$

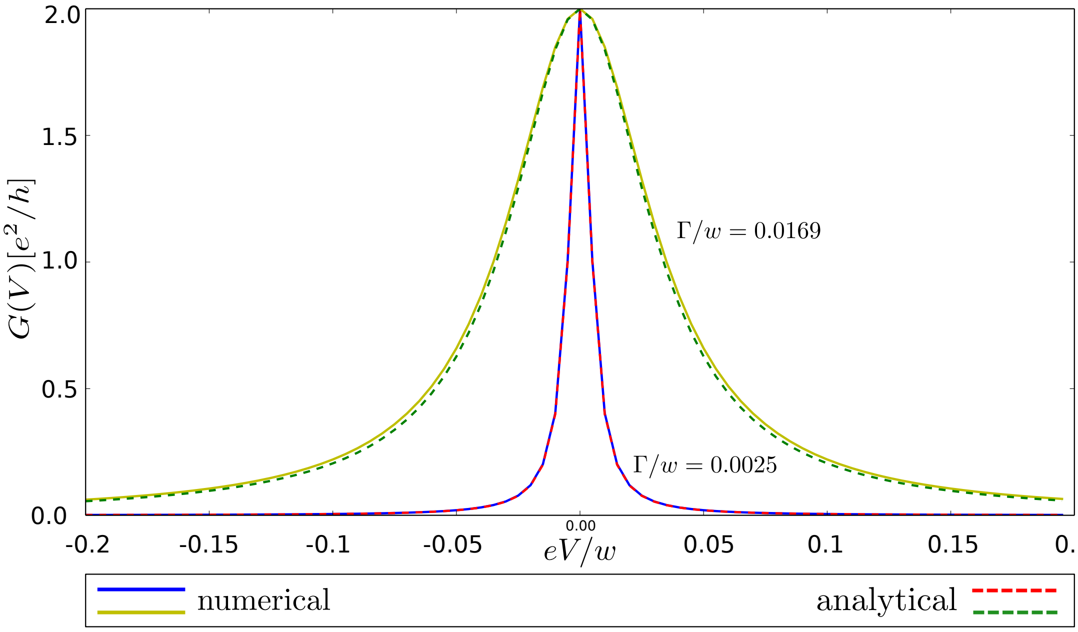

Figure 4: Conductance of a N-S junction in case of isolated Majorana states. The results of the numerical and the analytical calculation (dashed) for $\Gamma/w = 0.0025, 0.0169$ are presented. In both cases $\Delta/w = 1.0,\ \mu_{\rm N}/w = \mu_s/w = 0.0$ and $L = 50$.

Electrons near the Fermi energy have the highest probability of tunneling into the superconductor. Since $E = 0$ for $\epsilon_{\rm kin} = \mu_{\rm N}$, cf., Eq. (39), we insert $\rho_{\rm N}(E = 0)$ into Eq. (37). Taking into account that the chemical potential in the leads is held at $\mu_{\rm N} = 0$, we can write $\Gamma/w = \gamma^2/w^2$. We notice that the conductance $G(V)$ is a Lorentz function with a maximum of $2e^2/h$ at zero bias voltage and that Γ is a measure of the width. The result of the numerical and the analytical calculation for two different values of Γ are shown in Fig. 4. We see that the numerical result is closer to the analytical result for small values of eV/Δ because the analytical derivation does not take into account the tunneling of electrons to excitation states above the energy gap. The choice of the chain length L does not influence the results because the Majorana zero modes are located exactly at the both ends and can not interact. In order that the conductance reaches a value of $2e^2/h$, an electron at this energy has to be Andreev-reflected with probability one, as we can deduce from Eq. (34).

3.2.2 Weakly coupled Majorana bound states

If we assume a slight deviation from the specific case $\mu_s/w = 0$ and $|\Delta|/w = 1$, there will be an weak interaction between the Majorana states $\gamma_{L,R}$ of the two opposite sites of the superconductor as we have pointed out in section 2.2. This interaction can be described by the effective Hamiltonian Eq. (19) with the parameter $t = \omega e^{-L/l_0}$ that describes the exponentially decreasing coupling strength for increasing chain length L. This expression for t is exact in the limit $l_0/L \ll 1$. In case of our one-dimensional tight binding model the relation between the inverse length l_0^{-1} and the parameters of the system is presented in [18] and is given by

$$l_0^{-1} = \min(\big|\ln|x_+|\big|, \big|\ln|x_-|\big|)\,, \qquad \text{with} \quad x_\pm = \frac{-\mu_s \pm \sqrt{\mu_s^2 - 4w^2 + 4|\Delta|^2}}{2(w + |\Delta|)}. \tag{41}$$

An analytical formula for ω is not given. However, it should be slowly varying which means that it is a polynomial and not an exponential function of the system's parameters because exponentially terms are taken into account in l_0. The analytical formula for the conductance

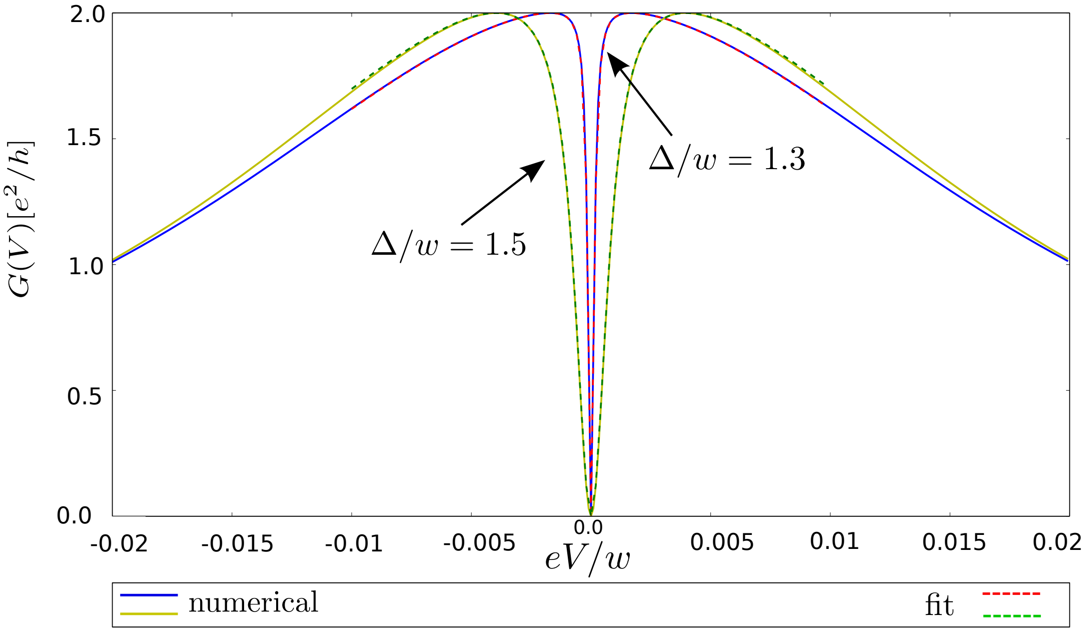

Figure 5: Conductance of a N-S junction in case of weakly coupled Majorana states. The results of the numerical calculation (dashed) and the fit of the analytical formula (42) for $\Delta/w = 1.5, 1.3$ are presented. The fit result is t $= 2.00 \cdot 10^{-3}, 0.85 \cdot 10^{-3}$. In both cases $\mu_{\rm N}/w = \mu_s/w = 0.0$, $L = 5$ and $\Gamma/w = 0.01$.

$G(V)$ is derived in [9] and is given by

$$G(V) = \frac{2e^2}{h} \frac{(4eV\Gamma)^2}{[(eV)^2 - 4t^2]^2 + (2eV\Gamma)^2}. \tag{42}$$

The conductance defined by Eq. (42) reproduces Eq. (36) for $t = 0$, whereas it goes to zero for $eV = 0$ and reaches $2e^2/h$ at $eV = \pm 2t$ in case of $t \neq 0$. Since we do not have any analytical relation between the parameters of the tight binding system and ω, we will have one free parameter if we compare the numerical with the analytical result. Therefore we fit (42) to the numerical results to determine t where we treat Γ as a free parameter, too, in order to find the best approximation for it. However, we find that $\Gamma/w \approx \gamma^2/w^2$ for all fits as we have derived in Eq. (36)). We restrict the fitting to the interval from $eV = -\Gamma$ to $eV = \Gamma$ because the approximation that Γ is independent of eV holds for small energies eV. Fig. 5 shows the result of the numerical calculation of the conductance and the fit result for two different values for Δ/w. If we assume that ω does not depend on L, we can compare the numerical and the analytical result of l_0^{-1} as follows. We keep μ_s/w and Δ/w constant and determine t for different chain lengths L. If we plot $\ln(t)$ versus L, we will obtain a linear relation of the form $\ln(t) = -l_0^{-1} L + \ln(\omega)$ where we can determine the slope l_0^{-1} and the y-intercept $\ln(\omega)$ via linear regression (cf. Fig. 6). We notice that the regression line is very close to the numerical results and the standard deviation of the slope and the y-intercept is small, not just in the case presented in Fig. 6, but as long as $l_0/L \ll 1$. This means that the assumption that ω is independent of L makes sense. Moreover, the determined values for l_0^{-1} fit to the analytical calculation (cf. Tab. 1).

Another method to study the relationship between t and the parameters of the tight binding system is, to keep L and w constant and vary Δ/w and μ_s/w. Wee would not obtain new, independent results by vary w as well because w defines the energy scale of the system and Δ and μ_s are measured in units of it. Since there is no analytical prediction for ω, we could have determined it for each given pair of values for Δ/w and mu_s/w by doing a linear regression for different lengths of the system. However, this procedure is very time-expensive so that we

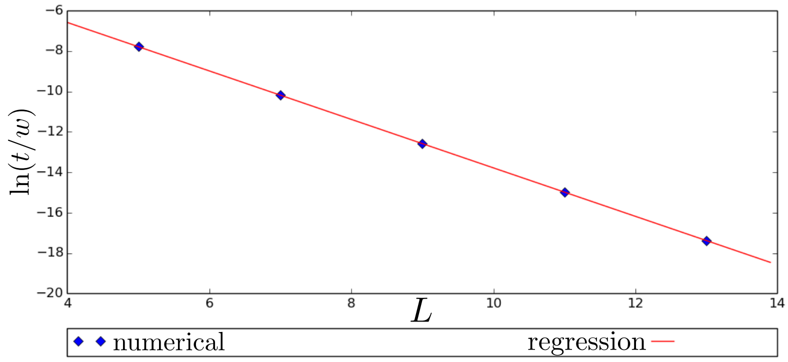

Figure 6: Plot of $\ln(t)$ vs. L and linear regression with all other parameters held constant, $\Delta/w = 1.2, \mu_s/w = 0.0, \Gamma/w = 0.01$. We obtain $l_0^{-1} = 1.19915(10)$, which fits to the analytical result $l_0^{-1} = 1.19895$. Due to the y-intercept we obtain $\omega = 0.16638(15)$. The largest value of l_0/L is 0.16.

Δ/w	μ_s/w	$\max(l_0/L)$	l_0^{-1}(num.)	l_0^{-1}(an.)	ω
1.1	0.1	0.14	1.4517(35)	1.413	0.1845(62)
1.1	0.2	0.15	1.3204(25)	1.306	0.1321(31)
1.2	0.0	0.16	1.19915(10)	1.19895	0.16638(15)
1.3	0.0	0.2	1.018484(23)	1.01844	0.138168(30)
1.25	0.1	0.18	1.09861(20)	1.032	0.149998(20)

Table 1: Comparison between the numerical (num.) and analytical (an.) result for l_0^{-1} for different values of Δ/w and μ_s/w with all other parameters held constant, $\Gamma/w = 0.01$. All calculations are done for chain lengths L in the range of 5 to 13. For values $L > 13$ the distance between the two conductance peaks becomes too small we are not able to determine a value for t which satisfy $t \neq 0$. The uncertainty in brackets refers to the last two digits of the value and arises from the linear regression.

choose one single point $(\Delta/w|\mu_s/w)$ where we calculate ω and take this value as a constant ω in the analytical calculation. Again, we determine t numerically by fitting Eq. (42) to the computed conductance. The result is shown in Fig. 7.

However, we have to emphasize that the zero bias voltage dip is very hard to resolve in an experimental setup. If we assume a one-dimensional superconducting wire made of 100 atoms ($L = 100$) and l_0^{-1} to be in order of 1, we will obtain $t \approx 10^{-44} eV$. With increasing temperature T the Fermi distributions, which can be taken as the integral limits in Eq. (34), become "smeared out" inside an interval of $\approx k_\mathrm{B}T$. Therefore we will expect that the currents and hence the conductance of a setup with $t = 0$ and one with $t \neq 0$ will differ measurably, if $k_\mathrm{B}T \approx t$. For the assumed parameters we find that we require a temperature of $\approx 10^{-46} K$ to be able to resolve the conductance dip at zero voltage which is quite unrealistic.

3.2.3 N-S-N junction

We consider a superconductor that is connected to two normal leads (R/L) with reservoirs held at electrochemical potentials $\tilde{\mu}_{\mathrm{L/R}}$. The approach to derive a formula for the currents $I_{\mathrm{L/R}}$ through the leads is quite similar to the procedure in case of the N-S junction. We measure $\tilde{\mu}_{\mathrm{L/R}}$ in reference to $\tilde{\mu}_s$, which we set to zero and split both leads into an electron and a hole lead, that are connected to the superconductor that is split in electron and hole part, too. This looks like the doubling of the N-S junction but we have to keep in mind that electrons and holes can cross the superconductor by entering at one side, then tunneling to the other side, and leaving the superconductor there. Therefore we have transmission coefficients $T^{L/R,R/L}_{\mathrm{Ne/h,\ Ne/h}}$

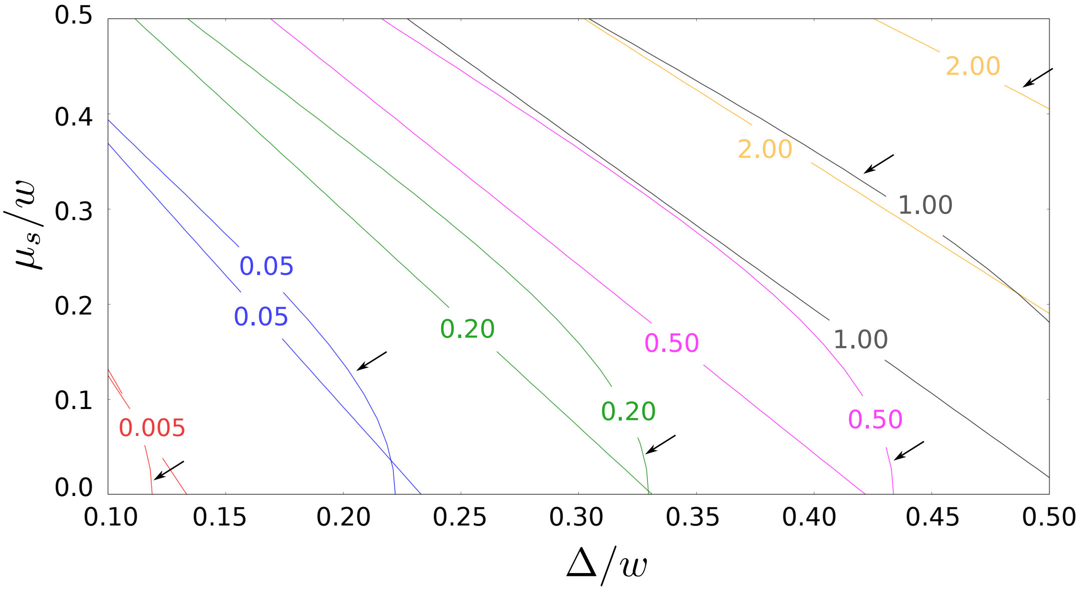

Figure 7: Comparison between the numerical (marked with an arrow) and analytical result for t/w as a function of Δ/w and μ_s/w. We assumed $\omega = 0.138$. For this choice the numerical and the analytical result are the same at point $(\Delta/w = 1.3|\mu_s/w = 0.0)$ (cf. Tab. 1). The other parameters are held constant at $\Gamma/w = 0.01$ and $L = 9$.

and $T^{L/R,R/L}_{\mathrm{Ne/h,\ Nh/e}}$ the latter describing electrons from different leads forming a Cooper pair and entering the superconductor. However, the probability of these processes dependent on the coupling strength between the Majorana states at the two sides and thus on the parameter t. In the following we consider the case of uncoupled Majorana states, hence $t = 0$, which we can achieve by choosing $\mu_s = 0$ and $w = |\Delta|$ or $L \gg 1$. Thus, the N-S-N junction can be taken as two independent N-S junctions and we can use Eq. (34) to determine the currents $I_{\mathrm{L/R}}$.

3.3 Floating superconductor

In this chapter we want to study the conductance of a floating superconductor that is connected to two normal leads. In the previous section we have considered a grounded superconductor and it is not obvious how to change the model that way that the superconductor becomes floating because then it has to preserve the particle number. But we saw that the grounding is an intrinsic property of the model of the superconductor because its Hamiltonian does not preserve the particle number due to the mean field approximation. A solution is to consider the currents that have to be conserved. In case of two leads (L/R) it means that the current I_{L} entering the superconductor through the left lead is the same as the current $-I_{\mathrm{R}}$ leaving through the right lead. Since $I_{\mathrm{L/R}}$ is a function of the applied voltage $V_{\mathrm{L/R}}$ thus of the chemical potential $\mu_{\mathrm{L/R}}$, we are able to determine $\mu_{\mathrm{R/L}}$ for a given value of $\mu_{\mathrm{L/R}}$ so that $I_{\mathrm{L}} = -I_{\mathrm{R}} = I$. Finally, we are able to calculate the conductance $G(V) = \frac{\partial I}{\partial V}$, where we define V as the voltage difference between the two leads. Hence $V = V_{\mathrm{L}} - V_{\mathrm{R}}$, taking the signs of $V_{\mathrm{L/R}}$ into account. Following the reasoning in section 3.2.3, we can use Eq. (34) to calculate the currents $I_{\mathrm{L/R}}$ and obtain

$$
\begin{aligned}
I_{\mathrm{L/R}} &= -\frac{1}{e}\int dE\, G_{\mathrm{L/R}}(E)\big[f_{\mathrm{L/R}}(E, V_{\mathrm{L/R}}) - f_0(E, V_s = 0)\big], \\
\text{with} \quad f(E,V) &= \frac{1}{1 + e^{\beta(E-eV)}},
\end{aligned}
\tag{43}
$$

where $f_{\mathrm{L/R}}$ is the Fermi distribution of the left/right lead and f_0 the one of the superconductor. $G_{\mathrm{L/R}}$ denotes the conductance which can be determined by solving the associated scattering problem. In section 3.2 we saw that the conductance is approximately given by a Lorentz function in case of $eV \ll \Delta$ and $L/l_0 \gg 1$, whose width depends on the coupling strength $\Gamma_{\mathrm{L/R}}$.

3.3.1 Symmetric N-S-N junction and finite temperatures

First, we will consider a symmetric N-S-N junction at low temperature ($\beta^{-1} = k_{\mathrm{B}}T \ll 1$) and $\Gamma_{\mathrm{L}}/w = \Gamma_{\mathrm{R}}/w = \Gamma/w$, hence $G_{\mathrm{L}}(E) = G_{\mathrm{R}}(E) = G(E)$. As we have pointed out, the current has to be conserved which leads us to

$$0 \stackrel{!}{=} I_{\mathrm{L}} + I_{\mathrm{R}} = -\frac{1}{e}\int dE\, G(E)\big[f_{\mathrm{L}}(E, V_{\mathrm{L}}) + f_{\mathrm{R}}(E, V_{\mathrm{R}}) - 2f_0(E, V_s = 0)\big], \tag{44}$$

where we now have to determine V_{R} as a function of V_{L} so that the equation is valid for a given V_{L}. Due to the system's symmetry we choose $V_{\mathrm{R}} = -V_{\mathrm{L}}$ and notice that the term in square brackets becomes antisymmetric, whereas $G(E)$ is symmetric in in E so that the integral is zero. Since the two currents $I_{\mathrm{L/R}}$ are identical apart from a sign, we focus on $I = I_{\mathrm{L}}$. To be able to solve the integral approximately for finite temperatures, we make use of the Sommerfeld expansion.

As an example, we consider the integral

$$I = \int dE\, G(E) f(E, V), \tag{45}$$

where we define the variables as we did in (43). Integration by parts leads to

$$I = -\int dE\, \tilde{G}(E)\frac{\partial f(E, V)}{\partial E} = \int dE\, \tilde{G}(E)\beta\frac{e^{\beta(E-eV)}}{(1 + e^{\beta(E-eV)})^2}, \tag{46}$$

where $\tilde{G}$ denotes the antiderivative of G and where we used that $G(\pm\infty) = 0$. We notice that the Fermi distribution is “smeared out” inside the interval $\approx eV \pm \beta^{-1}$ for $\beta^{-1} \ll 1$ (low temperatures). Therefore, the derivative $\partial f/\partial E$ is peaked at eV with a peak width $\approx 2\beta^{-1}$. The idea of the Sommerfeld expansion is that we can write $\tilde{G}$ as a Taylor series at $E = eV$ which leads to analytically solvable integrals. We limit ourselves to the first non-zero correction term. Therefore we have to assume that the first three terms of the Taylor series are sufficient to approximate $\tilde{G}$, as we will see later. Writing $\tilde{G}$ as a Taylor series at $E = eV$, we obtain

$$I = \int dx \left[\tilde{G}(eV) + \frac{\partial \tilde{G}(x + eV)}{\partial x}\bigg|_{x=0} x + \frac{1}{2}\frac{\partial^2 \tilde{G}(x + eV)}{\partial x^2}\bigg|_{x=0} x^2 + \mathcal{O}(x^3)\right]\beta\frac{e^{\beta x}}{(1 + e^{\beta x})^2}, \tag{47}$$

where we have to solve the integrals $I_i = \int dx\, x^i \beta \frac{e^{\beta x}}{(1+e^{\beta x})^2}$ for $i = 0, 1, 2$. For odd powers of x the integrand is an odd function so that these integrals are zero which is the reason why we need the second order term to calculate the first non-zero correction. The other integrals can be looked up in [28] where we find $I_0 = 1$ and $I_2 = \pi^2/3$. Finally we can rewrite Eq. (47) as

$$I = \int_{-\infty}^{eV} G(E)\, dE + \frac{\pi^2}{6}\frac{\partial G(E)}{\partial E}\bigg|_{eV}\beta^{-2} + \mathcal{O}(\beta^{-4}), \tag{48}$$

where we notice that the first term describes the case of zero temperature because then the Fermi distributions are Θ-functions and determine the integral limits. We can apply the result (48) to Eq. (43) to gain an approximative expression for the current $I(= I_{\mathrm{L}})$

$$I \approx -\frac{1}{e}\left[\int_0^{eV_{\mathrm{L}}} G(E)\, dE + \frac{\pi^2}{6\beta^2}\underbrace{\frac{\partial G(E)}{\partial E}\bigg|_{eV_{\mathrm{L}}}}_{G'(eV_{\mathrm{L}})} - \frac{\pi^2}{6\beta^2}\underbrace{\frac{\partial G(E)}{\partial E}\bigg|_{0}}_{=0}\right], \tag{49}$$

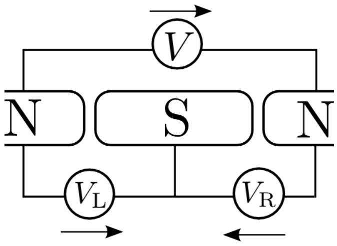

Figure 8: Sketch of the N-S-N junction taken as a voltage divider. If the two N-S junctions are identical as well as the currents, then $V_{\mathrm{L}} = -V_{\mathrm{R}}$ and $V = 2V_{\mathrm{L}}$. This means that the conductance measured with respect to V will be the half of the one with respect to V_{L}.

where we neglected terms in the order of β^{-4}. The last term will be zero, if the conductance can be described by a Lorentz function because it has a maximum at zero voltage.
Now we introduce the two new voltages

$$V = V_{\mathrm{L}} - V_{\mathrm{R}} \quad \text{and} \quad \alpha = \frac{V_{\mathrm{L}} + V_{\mathrm{R}}}{2}, \tag{50}$$

where V is the total voltage difference between the two leads which one would measure in an experimental setup. Therefore we derive a formula for the total conductance G_{tot} measured with respect to V. Solving (50) for V_{L} and V_{R} we obtain

$$V_{\mathrm{L}} = \alpha + \frac{V}{2} \quad \text{and} \quad V_{\mathrm{R}} = \alpha - \frac{V}{2}. \tag{51}$$

In our specific case $V = 2V_{\mathrm{L}}$ and $\alpha = 0$. This leads us to the total conductance

$$G_{\mathrm{tot}}(eV) = \frac{\partial I}{\partial V} = \frac{\partial I}{\partial(eV_{\mathrm{L}})} \underbrace{\frac{\partial(eV_{\mathrm{L}})}{\partial V}}_{=e/2} = \frac{1}{2}\Big[G(eV/2) + \frac{\pi^2}{6\beta^2} G''(eV/2)\Big]. \tag{52}$$

That means that, in case of zero temperature (and $\gamma \ll w$ and $eV \ll \Delta$), the total conductance of a symmetric N-S-N junction is the half of the conductance of a single N-S junction. Especially, assuming that the single lead conductance is given by the Lorentz function (34) it means that G_{tot} reaches $\frac{e^2}{h}$ at its maximum. Due to the Landauer-Büttiker-formalism, this result is not surprising because the system can be taken as a voltage divider, cf. Fig. 8. The numerical calculation via *Kwant* confirms Eq. (52) in case of zero temperature, cf. Fig. 9, as well as in case of small temperatures $\beta^{-1} \ll \Delta$, cf. Fig. .10. If the conductance is given by Eq. (36), we obtain

$$G''(eV) = -\frac{16\Gamma^2\big[-3(eV)^2 + 4\Gamma^2\big]}{\big[4\Gamma^2 + (eV)^2\big]^3} \tag{53}$$

for the second derivative. It is negative for an interval of $eV = \pm 2\Gamma/\sqrt{3}$ around zero and reaching $-1/\Gamma^2$ at $eV = 0$ so that the conductance peak decreases with increasing temperature $\propto \Gamma^{-2}\beta^{-2}$.

3.3.2 Asymmetric N-S-N junction

Now we take a look at an asymmetric N-S-N junction with $\Gamma_{\mathrm{L}} \neq \Gamma_{\mathrm{R}}$. Again we search for an expression for the total conductance by equating I_{L} with $-I_{\mathrm{R}}$ and introducing the new variables V and α defined in Eq. (50). In general, we are not able to give an analytical relationship between V_{L} and V_{R} so that $g(V_{\mathrm{L}}, V_{\mathrm{R}}) = I_{\mathrm{L}} + I_{\mathrm{R}} = 0$ because in case of non-zero temperature the occurring integrals are not analytically solvable. Therefore we assume that the constraint $g = 0$ is fulfilled for the voltages V_{L}^0 and V_{R}^0 thus $g(V_{\mathrm{L}}^0, V_{\mathrm{R}}^0) = 0$. Our aim is to determine

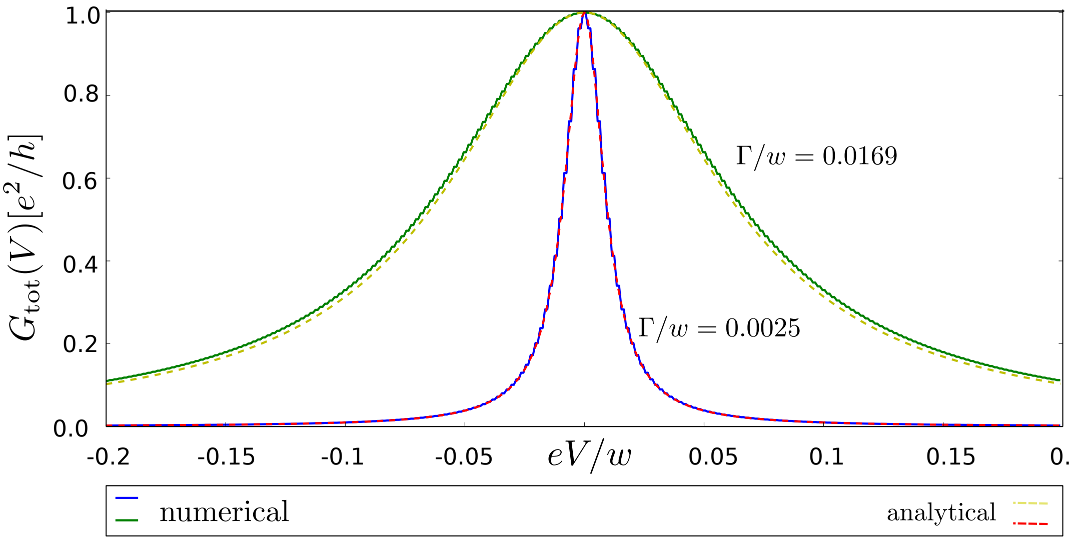

Figure 9: The conductance of a N-S-N junction in case of $\Gamma_\mathrm{L} = \Gamma_\mathrm{R} = \Gamma$ and zero temperature. Compared to the N-S junction with isolated Majorana modes (cf. Fig. 4), we see that the conductance is halved and reaches e^2/h at its maximum. Notice that the energy is expressed in terms of eV, whereas it is expressed in terms of eV_L in Fig. 4. Again we see that the difference between the analytical and the numerical result is smaller for small values of Γ. All other variables are held constant at $\Delta/w = 1.0$ and $L = 50$.

$G_\mathrm{tot}(eV) = \left(\frac{\partial I}{\partial V}\right)_g\Big|_{(V_\mathrm{L}^0, V_\mathrm{R}^0)}$ which means that we differentiate I with respect to V while keeping g constant and evaluate this expression at the point $(V_\mathrm{L}^0, V_\mathrm{R}^0)$ afterwards. We focus on the left current ($I = I_\mathrm{L}$) and mind that I and g are functions of V and α so that their total differentials are given by

$$dI(V,\alpha) = \left(\frac{\partial I_\mathrm{L}}{\partial V}\right)_\alpha dV + \left(\frac{\partial I_\mathrm{L}}{\partial \alpha}\right)_V d\alpha \quad \text{and} \quad dg(V,\alpha) = \left(\frac{\partial g}{\partial V}\right)_\alpha dV + \left(\frac{\partial g}{\partial \alpha}\right)_V d\alpha. \tag{54}$$

Taking into account that $g = I_\mathrm{L} + I_\mathrm{R}$, we can rewrite $\left(\frac{\partial I}{\partial V}\right)_g$ as

$$\left(\frac{\partial I}{\partial V}\right)_g = \left[\left(\frac{\partial I_\mathrm{L}}{\partial V}\right)_\alpha \left(\frac{\partial I_\mathrm{R}}{\partial \alpha}\right)_V - \left(\frac{\partial I_\mathrm{L}}{\partial \alpha}\right)_V \left(\frac{\partial I_\mathrm{R}}{\partial V}\right)_\alpha\right] \left[\left(\frac{\partial I_\mathrm{L}}{\partial \alpha}\right)_V + \left(\frac{\partial I_\mathrm{R}}{\partial \alpha}\right)_V\right]^{-1}. \tag{55}$$

Now we can express the derivatives of the current with respect to V and α in terms of derivatives with respect to $V_\mathrm{L}, V_\mathrm{R}$ where we can use that $\partial V_{\mathrm{L/R}}/\partial V = \pm 1/2$ and $\partial V_{\mathrm{L/R}}/\partial \alpha = 1$ which follows from Eq. (51). Moreover, $I_{\mathrm{L/R}}$ does not depend on $V_{\mathrm{R/L}}$. Therefore Eq. (55) reduces to

$$\left(\frac{\partial I}{\partial V}\right)_g = \left(\frac{\partial I_\mathrm{L}}{\partial V_\mathrm{L}}\right)\left(\frac{\partial I_\mathrm{R}}{\partial V_\mathrm{R}}\right) \left[\left(\frac{\partial I_\mathrm{L}}{\partial V_\mathrm{L}}\right) + \left(\frac{\partial I_\mathrm{R}}{\partial V_\mathrm{R}}\right)\right]^{-1}. \tag{56}$$

Up to this point the derivation is valid even for non-zero temperatures. If the single lead conductances are given by Lorentz functions, the occurring integrals will not be solvable analytically. Therefore we limit ourselves to the first term of the Sommerfeld expansion, hence $T = 0$, to give an analytical solution. Taking into account that $\partial I_{\mathrm{L/R}}/\partial V\mathrm{L/R} = G_{\mathrm{L/R}}(eV_{\mathrm{L/R}})$, we can write

$$G_\mathrm{tot}(eV) = \left(\frac{\partial I}{\partial V}\right)_g\Bigg|_{(V_\mathrm{L}^0, V_\mathrm{R}^0)} = \frac{G_\mathrm{L}(eV_\mathrm{L})\, G_\mathrm{R}(eV_\mathrm{R})}{G_\mathrm{L}(eV_\mathrm{L}) + G_\mathrm{R}(eV_\mathrm{R})}\Bigg|_{(V_\mathrm{L}^0, V_\mathrm{R}^0)}, \tag{57}$$

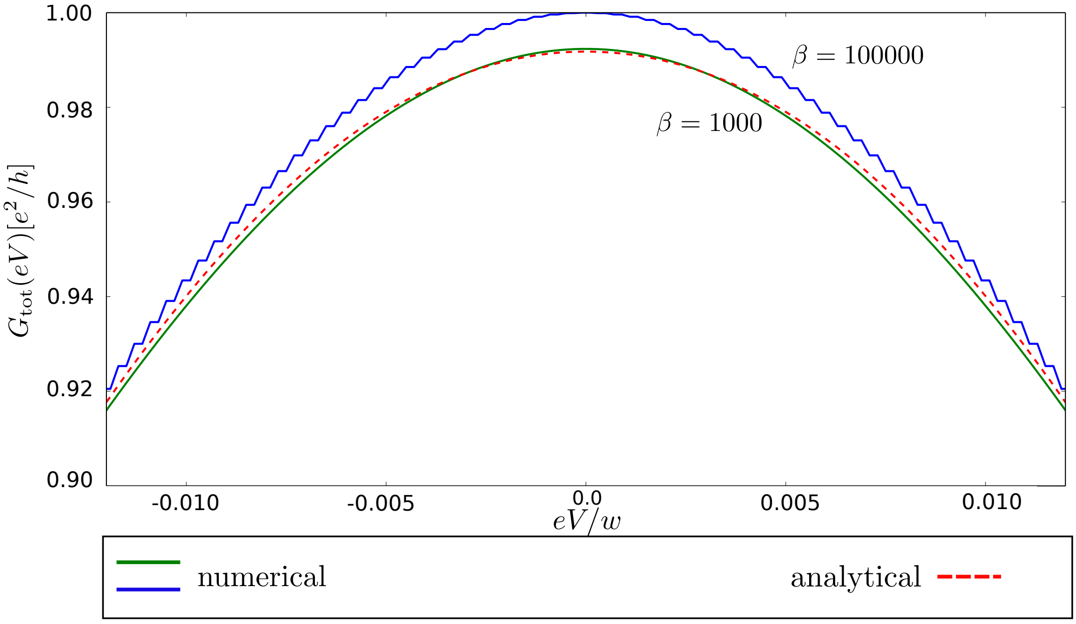

Figure 10: Comparison of the conductance of a symmetric N-S-N junction for zero temperature (approximated by $\beta^{-1}/w = 10 \cdot 10^{-5}$) and for small temperatures ($\beta^{-1}/w = 10^{-3}$). For the latter the analytical result calculated by using Eq. (52) is presented (dashed). $\Delta/w = 1$ and $L = 50$ in both cases. The conductance peak decreases with increasing temperatures $\propto \Gamma^{-2}\beta^{-2}$. For the given values, the difference at $eV = 0$ is $0.0082e^2/h$.

where $V_{\rm L}^0$ and $V_{\rm R}^0$ are functions of V, whereas α is determined by the constraint $g(V, \alpha) = 0$. We see that Eq. (57) will reproduce Eq. (52) if we assume that $\Gamma_{\rm L} = \Gamma_{\rm R}$ and therefore $V_{\rm L} = -V_{\rm R}$. It remains to derive the relation between $V_{\rm L}$ and $V_{\rm R}$ in case of $\Gamma_{\rm L} \neq \Gamma_{\rm R}$ (and still $T = 0$). Moreover $G_{\rm L/R}$ is given by the Lorentz function (36) so that the constraint $g = 0$ can be written as

$$g(V_{\rm L}^0, V_{\rm R}^0) = I_{\rm L} + I_{\rm R} = 0 \;\Leftrightarrow\; \frac{2e}{h}\int_0^{eV_{\rm L}^0} dE\, \frac{4\gamma_{\rm L}^4}{4\gamma_{\rm L}^4 + E^2} = -\int_0^{eV_{\rm R}^0} dE\, \frac{4\gamma_{\rm R}^4}{4\gamma_{\rm R}^4 + E^2}. \tag{58}$$

Taking into account that the antiderivative of $(1 + x^2)^{-1}$ is given by $\arctan(x)$, we can solve this expression for $eV_{\rm R}$ and obtain

$$eV_{\rm R}^0 = -2\gamma_{\rm R}^2 \,\tan\left[\left(\frac{\gamma_{\rm L}}{\gamma_{\rm R}}\right)^2 \arctan\left(\frac{eV_{\rm L}^0}{2\gamma_{\rm L}^2}\right)\right], \tag{59}$$

where we can always choose $\Gamma_{\rm L} < \Gamma_{\rm R}$ because we can decide which site is the left/right site. Therefore Eq (59) is continuously differentiable at every point and goes to $\mp 2\gamma_{\rm R}^2 \tan\left[(\frac{\gamma_{\rm L}}{\gamma_{\rm R}})^2 \frac{\pi}{2}\right]$ for $eV_{\rm L} = \pm\infty$. We notice that it will reproduce the relation $V_{\rm R} = -V_{\rm L}$, if we set $\gamma_{\rm L} = \gamma_{\rm R}$ as we did in the symmetric case. Moreover, the function passes through the origin (cf. Fig. 11) which means that $V_{\rm L}^0 = V_{\rm R}^0 = 0$ is a solution to the constraint $g = 0$. Using Eq. (50), we find that at this point $V = 0$, too. Thus we can rewrite (57) as

$$G_{\rm tot}(eV = 0) = \frac{G_{\rm L}(0)\, G_{\rm R}(0)}{G_{\rm L}(0) + G_{\rm R}(0)} = \frac{e^2}{h} \tag{60}$$

because the single lead conductance peaks at zero voltage and reaches $2e^2/h$. From Eq. (57) we can deduce that the total conductance will reach its maximum if the single lead conductances reach theirs. Hence the zero temperature conductance of a N-S-N junction in case of uncoupled Majorana states and spinless electrons peaks at $V = 0$ and takes the value of e^2/h. Due to the Landauer-Büttiker formalism, we can interpret Eq. (57) as the result of the voltage divider shown in Fig. 8. If we assume that the voltages $V_{L/R}$ arise from resistances $R_{L/R} = G_{L/R}^{-1}$, we can write the total voltage V as $V = V_L - V_R$ and obtain

$$\frac{I}{G_{tot}} = \frac{I_L}{G_L} - \frac{I_R}{G_R} \Leftrightarrow G_{tot} = \frac{G_L G_R}{G_L + G_R}, \tag{61}$$

where we used $I = I_L = -I_R$. This is exactly the form of Eq. (57).

Of course we can also determine the relationship between V_L^0 and V_R^0 numerically. For a given value for V, we estimate α and calculate the currents $I_{L/R}$. Depending on which of them is grater, we modify α and repeat the procedure until the difference between $I_{L/R}$ drops below a given threshold. At this point, we can solve V and α for $V_{L/R}$ and compare the result with Eq. (59) (cf. Fig. 11).

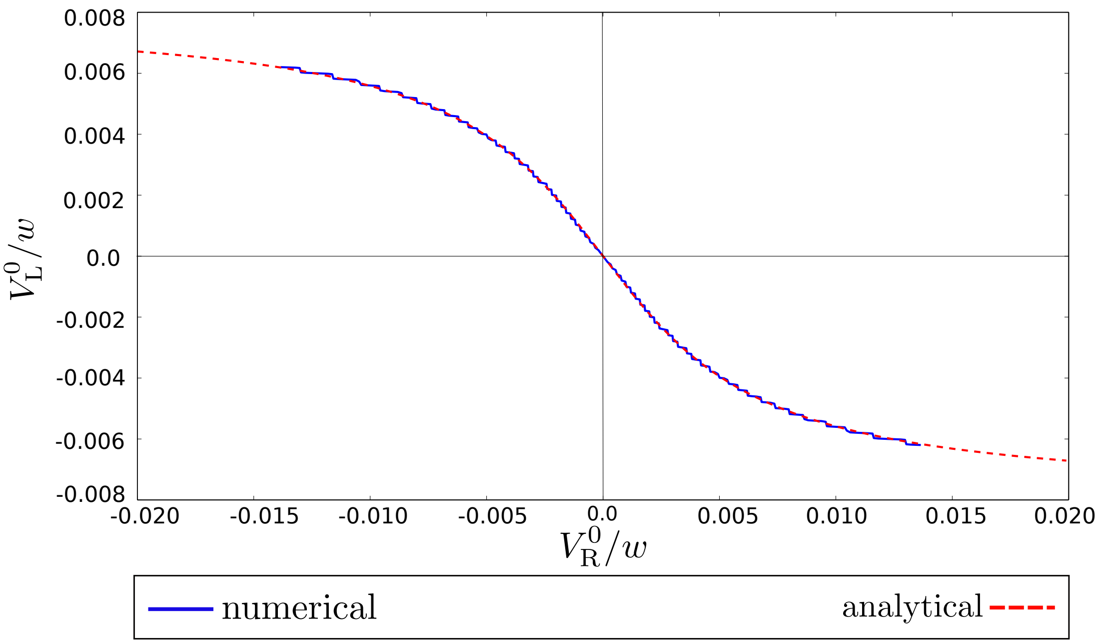

Figure 11: Comparison of the numerical and the analytical calculation of the relation between V_L^0 and V_R^0 that fulfill $g(V_L^0, V_R^0) = 0$ so that the current through the superconductor is preserved. The parameters are set to $\Gamma_L/w = 0.0025$, $\Gamma_R/w = 0.0169$, $\Delta/w = 1.0$,and $L = 100$.

4 Conclusion / Outlook

At this point we can summarize and compare the results of the different systems we have considered. We have made use of the Landauer-Büttiker formalism (LB) to describe the transport of spinless, non-interacting electrons through one-dimensional junctions between normal, ideal leads and a superconductor with Majorana bound states at its ends in two different cases: On the one hand, the N-S junction where the superconductor is grounded and on the other hand, the N-S-N junction with a floating superconductor. Due to the numerical and analytical calculations, we can determine the differential conductance and especially its maximum value. It turn out that the zero bias conductance at zero temperature is halved form $\partial I/\partial V = 2e^2/h$ in case of a grounded to $\partial I/\partial V = 1e^2/h$ in case of a floating superconductor which is the main result of this work.

In the context of LB this conductance halving appears in a very intuitive way because we can take both N-S contacts of the N-S-N junction as an origin of electrical resistance and gain a voltage divider.

From another point of view, the conductance halving seems to be more surprising. If we compute the maximal conductance of an normal, ideal conducting sample between two normal leads (N-N-N), we will find that the conductance reaches e^2/h. It is exactly the result of a junction between two normal, ideal leads (N-N) we have had a quick glance at to determine the conductance quantum. This gives rise to the question why does the halving appear for N-S-N but not for N-N-N junctions. A possible answer is to focus on the leads and the reservoirs. We notice that the grounded superconductor is the only "lead" which can transport two electrons at the same time in form of Cooper pairs to a reservoir of Cooper pairs. However, in the other cases the sample is connected to two normal electron reservoirs by normal leads. These are not able to transfer Cooper pairs.

A N-S-N junction very similar to ours but with strong electron-electron interaction is studied by L. Fu in [11]. He considers a floating superconductor with Majorana bound states which which are spatially far apart from each other so that their overlap goes to zero. The superconductor is in addition connected to a capacitor of capacity C to whose other contact a gate voltage V_{g} is applied. By tuning V_{g} one can influence the number of electrons in the superconductor. The capacitor defines a charging energy $E_{\mathrm{C}} = e^2/(2C)$ which is the energy cost to increase or decrease the number of electrons by in the superconductor by one. This leads to a Coulomb interaction between the electrons. Whereas we have studied the systems for zero charging energy, L. Fu considers the case of great charging energy $E_{\mathrm{C}}/\Gamma \gg 1$. However, he obtains the same value of e^2/h for the maximum conductance. This leads us to the conjecture that the conductance peak of a floating superconductor reaches e^2/h independent of the charging energy and for spinless electrons.

Our result of conductance halving is incompatible with the work of R. Hützen et al. [14]. They consider a N-S-N junction with charging energy and obtain a continuous transition from $2e^2/h$ to e^2/h as the ratio E_{C} is increased form zero to strong charging energy. They take the value of $2e^2/h$ as a sign of Andreev reflection which is suppressed for $E_{\mathrm{C}}/\Gamma \gg 1$. In our case we have taken the N-S-N junction as a composition of two independent N-S junctions and we have found the conductance by equating the currents through these two junctions. Since we have assumed that Andreev reflection happens in a single NS junction, we can conjuncture that this does not change for the N-S-N junction even though the conductance halves.

Furthermore, we have to take a look on the result of the conductance of the N-S junction. The relation between conductance and bias voltage determined via LB and numerical solution of the scattering problem, agrees with the formula derived via Green's function method by K. Flensberg in [9]. For a superconductor with isolated Majorana states we obtain the predicted Lorentz function whose width depends on the tunneling rate $\Gamma/\hbar$ but whose maximum always is $2e^2/h$ at zero bias voltage and zero temperature. In contrast, the conductance peak will split into two maximums each reaching $2e^2/h$ while a dip to zero appears in between, if we assume weakly coupled Majorana states. The parameter t that describes the coupling strength and to which the width of the splitting is proportional [9], is derived by A. Y. Kitaev in [18]. Our results for t depending on the gap energy, the chemical potential of the superconductor, and its length coincide with Kitaev's work. However, we have pointed out that it will be very difficult to determine t experimentally by measuring the splitting of the conductance peak because on the one hand t decreases exponentially with the length and on the other hand it requires unaccessible low temperatures to resolve the dip.

Additionally, we have applied a Sommerfeld expansion to the conductance of the symmetric N-S-N junction where we have obtained that the conductance peak decreases proportional to $\Gamma^{-2}\beta^{-2}$ for increasing temperatures which we have confirmed numerically.

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