1) The Toric Code Model

Stabilizers on 2D lattice:

\[ \text{qubit} \]

\[ \text{periodic bond conditions} \]

\[ (= \text{torus}) \]

Stabilizers:

- For each vertex \( v \): \( A_v = 2^{E_v} \)
- For each plaquette \( p \): \( B_p = X^{E_p} \)

Note: \( [A_v, A_{v'}] = [B_p, B_{p'}] = [A_v, B_p] = 0 \)

What is \( \text{value of} \ 1 \ \text{eigenstate) } \text{(stabilized subspace)} \)?

Denote \( |0\rangle \) as \( - \) \( \text{("no line")} \)

Denote \( |1\rangle \) as \( - \) \( \text{("line")} \)
+1 state for $A_v$:

$\Rightarrow$ closed loop patterns:

$\begin{array}{cccc}
+ & - & - & - \\
\end{array}$

+1 state for $B_p$:

$\Rightarrow \text{wedge must be eq. for flipping any plaquette}$:

$\begin{array}{cccc}
\end{array}$

Possible joint +1 eigenstate:

Eq. right. Superposition of all loop patterns on torus.

Dimension of logical space?

$\begin{array}{cccc}
\end{array}$

$p/p \text{ pair: 2 qubits, 2 steckets}$

$\Rightarrow$ original "code space"?

$\begin{array}{cccc}
\end{array}$

Eq. 7: $\prod_v A_v = 1$ and $\prod_p B_p = 1$

(and these are the only constraints.)

$\Rightarrow N \text{ qubits, } N - 2 \text{ constraints } \Rightarrow 2 \text{ logical qubits}$!
Logical operators: \((\not\equiv S, \not\equiv \text{ commun. } u/S)\):

\[
X_u = \begin{array}{ccccccc}
\times & \times & \times & \times & \times & \times \\
\end{array}
\text{around torus}
\]

\[
Z_u = \begin{array}{cccc}
\times & \times & \times & \times \\
\end{array}
\]

\[
Z_v = \begin{array}{c}
\times \\
\times \\
\times \\
\times \\
\times \\
\end{array}
\]

\[
\frac{\alpha}{\beta}, \quad X_v, \quad \text{comm. u. all } A, B
\]

\[
\left[ Z_u, X_v \right] = 0; \quad \left[ Z_v, X_v \right] = 0, \quad \text{and}
\]

\[
Z_u X_v = -X_v Z_u, \quad Z_v X_u = -X_u Z_v
\]

\[
\Rightarrow (Z_u, X_v) \neq (Z_v, X_u) \text{ define logical gedichte.}
\]

Are there more logical gedichte?

\[\Rightarrow \text{No! Multiplication } u/A \not\equiv B \text{ can defrom loops, create local loop, or create an even } \# \text{ of loops around the torus, but cannot change the parity of loops around the torus.}\]
Topological spectra are topological!

Interpretation of code space in terms of loop patterns:

$\pm 1$ ambiguity of $2\pi$: even/odd # of loops cross $2\pi$

$\Rightarrow$ even/odd # of topologically non-trivial loops in vertical direction.

Same in horizontal direction.

$\Rightarrow$ 4 code states distinguished by global (topological) feature - locally states look identical!

Effect of local errors?

Example 2:

$P_1 \neq P_2$ $\Rightarrow$ antic. $U / B_1, B_2$

$\Rightarrow$ error syndrome on $P_1 \& P_2$. 

$\Rightarrow$ error syndrome on $P_1 \& P_2$. 

Can we "remove" error on eq. $R_2$?

→ apply $\text{l}_2$ around $R_2$:

$$P_1 \times P_2 \times P_3 \rightarrow \text{even, w/ } R_1, R_2$$

etc... → 2 errors can form strips (for dual lattice)

1) syndromes at the endpoints (= plaquettes)

2) endpt. = syndrome

Correction: undo string of 2 endpoints!

But: Syndrome does not reveal string (degree, code):

→ Error. Correction = top. third loop $\in S$

→ Error corrected!
Possible source of error: Pair of errors (a story)

has propagated very far & we pair them up around the torus ⇒ logical error!

⇒ impressed exponentially in size of torus!

Information topologically protected: Encoded in global property, thus stable against local errors)

X errors: Similarly, but among original lattice & endpoints at vertices.

⇒ Topological quantum memory!
2. Toric Code as a topological model

Define 2D spin model (same lattice) as

\[ H = -\sum_v A_v - \sum_p B_p \]

- ground space = code space of TC.
- elementary excitations = strings of 2's (or X's) with 2 endpoints.
- Excitations = endpoints, \( \equiv \) energy > 0

\[ \Rightarrow \text{excitations come in pairs!} \]

Creation / annihilation of excitations by eaching a 2 (or X):

\[ \ldots \rightarrow \begin{array}{c}
2 \end{array} \rightarrow \begin{array}{c}
2 \end{array} \rightarrow \begin{array}{c}
2 \end{array} \rightarrow \ldots \text{ etc.} \]

\[ \Rightarrow \text{annihil.} \]
What is the statistics of excitations?

1. **Self-statistics:**
   
   \[
   \begin{array}{c}
   \text{II vs. X} \\
   \text{II vs. X}
   \end{array}
   \]

   \[= \text{equal!}\]

   Same for X strings:

   \[= \text{Rosenz self-statistic!}\]

2. **Mutual statistics:**

   \[\text{create a loop, move in circle, accumulate.}\]

   \[
   \begin{array}{c}
   \text{2 Loop vanishes;}
   \\
   \text{2 Loop can be moved through X string and the vanishes:}
   \end{array}
   \]

   \[= \text{from } x^2 = -2x,\]

   \[\Rightarrow \text{preserves mutual statistics!}\]
(3) Combined particle of one X and one Y has fermionic statistics!

Emergent fermions in a non-commutative model!

More complex spin models:

Can have anyonic excitations
(with arbitrary braiding phases e^{i\varphi})

and even non-abelian anyons (with statistics described by matrices)

Emergent quasi-particles of anyonic states
(by condensing electrons + flux quanta).

Can be used for q. computation. Breital excitation & compute using the non-abelian braiding "phase" (can be read out interferometrically).

Q. Information stored in global state = topologically protected.