Multiport Impedance of the parity measurement network with Tunable Coupling Qubits

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Contents

1. Introduction

Qubits play an important role in quantum computation and quantum information. Promising candidates for quantum information processing are superconducting circuits, which usually involve Josephson Junctions that provide the necessary nonlinearity. An advantage of superconducting qubits is that we have the chance to build them ourselves and therefore to choose parameters at our own will, while we do not have such a flexibility in cavity Quantum Electrodynamics (QED) [1,2].

Quantum error correction is also essential for realizing fault tolerant quantum computing [3]. In error correction, checking parity of the qubits is fundamental in a sense that it enables us to detect if errors occur or not [4]. One of the approaches to perform stabilizer measurements requires a quantum circuit with a series of the qubits. In this method, the information is encoded in the state of ancilla qubits and then read out. Another approach proposed in [5] consists in directly measuring the parity of qubits by means of microwave techniques without the need of a quantum circuit. This scheme is further studied and adopted in [6]. In [6], taking advantage of what is called a Tunable Coupling Qubit (TCQ), well studied and introduced by [7], any intermediate quantum circuit is removed.

In this paper, I analyze the model in [6] with a so called 'black box' model and derive a multiport impedance, that gives the response of linear, passive, lossless and reciprocal systems using a method described in [8].

This paper begins with a brief review of the TCQ, which is followed by the necessary parity measurement condition to be met. In chapter 3, we introduce the circuit model of the measurement network and study two cases; the circuit with and without a drive port. Finally, I draw the conclusion in chapter 4

2. Parity Measurement Scheme

In this section, we briefly introduce the TCQ, which is exploited for parity measurement. We also give a review of parity measurement conditions proposed in [6].

2.1 Tunable Coupling Qubit (TCQ)

The TCQ that originally Gambetta et al. proposed consists of two transmons coupled by a capacitance, illustrated in Figure. 1 [7]. The idea of the TCQ is essentially that we can obtain more flexible qubit than a sole transmon, by encoding the qubit in the first two energy levels.

The Hamiltonian of this circuit can be derived by using the method in [9,10] and reads

$$H_{\mathrm{TCQ}} = \sum_{\pm} 4E_{C\pm}\left(n_{\pm} - n_{g\pm}\right)^2 - \sum_{\pm} 4E_{J\pm}\cos(\varphi_{\pm}) + 4E_I\left(n_{+} - n_{g+}\right)\left(n_{-} - n_{g-}\right) \quad (1)$$

where the symmetric case is assumed in which each capacitance of $\pm$ has the same value, $E_{C+} = E_{C-} = E_C = e^2\,(C_I - C_\Sigma)/\left[2\left({C_\Sigma}^2 + 2C_I C_\Sigma\right)\right]$ and $E_I = -2E_C\,C_I/(C_I + C_\Sigma)$, with $C_\Sigma = C + C_{\mathrm{g}}$. Then, we expect that in the so-called transmon regime where $E_{J\pm}/E_{C\pm}$ is adequately large, the energy levels do not depend on the reduced gate charge $n_{g\pm}$, resulting in the charge insensitivity as a simple transmon shows.

Moreover, by introducing creation and annihilation operators for each transmon mode $b_{\pm}^{\dagger}$ and $b_{\pm}$ respectively, expanding the cosine term up to fourth order in $\varphi_{\pm}$ and neglecting fast rotating terms, the Hamiltonian is second quantized

$$H_{\mathrm{TCQ}eff} = \sum_{\pm} \omega_{\pm} b_{\pm} b_{\pm}^{\dagger} + \frac{\delta_{\pm}}{2} b_{\pm}^{\dagger} b_{\pm}^{\dagger} b_{\pm} b_{\pm} + J\left(b_{\pm} b_{\pm}^{\dagger} + b_{\pm}^{\dagger} b_{\pm}\right) \quad (2)$$

where $\omega_{\pm} = \sqrt{8E_{J\pm}E_{C\pm}} - E_{C\pm}$, $\delta_{\pm} = -E_{C\pm}$, $J = 1/(\sqrt{2})\,E_I\left(E_{J+}E_{J-}/E_{C+}\,E_{C-}\right)^{1/4}$, and commutation relation $\left[b_{\pm}, b_{\pm}^{\dagger}\right] = 1$.

Furthermore, we approximately diagonalize the Hamiltonian by means of the unitary transformation

$$U_{\mathrm{TCQ}} = \exp\left[\lambda\left(b_{\pm} b_{\pm}^{\dagger} - b_{\pm}^{\dagger} b_{\pm}\right)\right] \quad (3)$$

with

$$\lambda = 1/2\,\arctan(-2J/\zeta) \quad (4)$$

where we denote ζ as $\omega_{+} - \omega_{-} - 2(\delta_{+} - \delta_{-})$. As a result of this unitary transformation, we obtain the effective Hamiltonian

$$\widetilde{H}_{\mathrm{TCQ}eff} = U_{\mathrm{TCQ}} H_{\mathrm{TCQ}} U_{\mathrm{TCQ}}^{\dagger} \approx \sum_{\pm} \widetilde{\omega}_{\pm} \tilde{b}_{\pm}^{\dagger} \tilde{b}_{\pm} + \frac{\tilde{\delta}_{\pm}}{2} \tilde{b}_{\pm}^{\dagger} \tilde{b}_{\pm}^{\dagger} \tilde{b}_{\pm} \tilde{b}_{\pm}^{\dagger} + \tilde{\tilde{\delta}}_{c} \tilde{b}_{+}^{\dagger} \tilde{b}_{+} \tilde{b}_{-}^{\dagger} \tilde{b}_{-} \quad (5)$$

where $\widetilde{\omega}_{\pm} = (\omega_{+} + \omega_{-})/2 \pm (\omega_{+} - \omega_{-})\cos(2\lambda)/2 \mp J\sin(2\lambda)$, $\tilde{\tilde{\delta}}_{\pm} =$

$(\delta_+ + \delta_-)\big(1 + \cos^2(2\lambda)\big)/2 \pm (\delta_+ - \delta_-)\cos(2\lambda)/2$ and $\tilde{\tilde{\delta}}_c = (\delta_+ + \delta_-)\sin^2(2\lambda)/2$. Here, we assume that $\delta_\pm$ is much smaller than $(\omega_+ - \omega_-)$ [6].

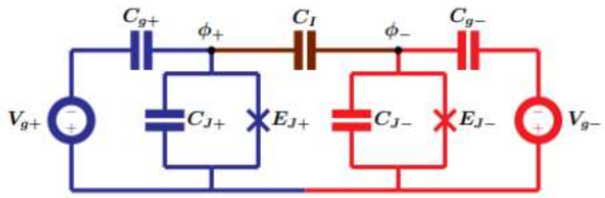

Figure. 1 Circuit model of the TCQ (used from [6])

2.2 Parity Measurement condition

In 2.2a, I review the general parity measurement condition of 3 qubits with two resonators, followed by the conditions for the system including TCQs.

2.2a Parity Measurement Condition of 3 Qubits

In the scheme in [6], the system is composed of three qubits coupled to two resonators. The crucial point is that only two bosonic modes are directly coupled to the common bath. Thus, the generic dispersive Hamiltonian of this system is written as

$$H = \sum_{i=1}^{3} \frac{\Omega_i}{2}\sigma_i^z + \left(\omega_1 + \chi_1 \sum_{i=1}^{3} \sigma_i^z\right) a_1^\dagger a_1 + \left(\omega_2 + \chi_2 \sum_{i=1}^{3} \sigma_i^z\right) a_2^\dagger a_2 + \chi_{12} \sum_{i=1}^{3} \sigma_i^z \left(a_1 a_2^\dagger + a_1^\dagger a_2\right) \quad (6)$$

Here, note that the qubit-qubit coupling is neglected because we assume the qubits to be off resonant. There appear not only qubit-state dependent dispersive shifts χ_1 and χ_2, but also qubit-state dependent coupling of the two resonators χ_{12}. This term emerges in the dispersive regime of multi-mode Jaynes-Cummings model, where detuning between the qubit frequency and the resonator frequency is much smaller than the coupling constant. Although some paper suggests that this term act as a quantum switch, turning on and off the interaction between two cavity modes, it has a significant effect on the parity measurement and is not negligible when assuming that the resonators' frequencies are not far away from each other [6]. In this system, the evolution of the output field depends on the Hamming weight by setting equal dispersive shift for each TCQ and the quantum switch.

In addition, by introducing input-output theory of two resonators, following [10], the evolution of each annihilation operator dependent on the Hamming weight read

$$\frac{da_{1,h_w}}{dt} = -i[\omega_1 + \chi_1(3-2h_w)]a_{1,h_w} - \chi_{12}(3-2h_w)a_{2,h_w} - \frac{\kappa_1}{2}a_{1,h_w} - \frac{\sqrt{\kappa_1\kappa_2}}{2}a_{2,h_w} - \sqrt{\kappa_1}b_{in} \quad (7)$$

$$\frac{da_{2,h_w}}{dt} = -i[\omega_2 + \chi_2(3-2h_w)]a_{2,h_w} - \chi_{12}(3-2h_w)a_{1,h_w} - \frac{\kappa_2}{2}a_{2,h_w} - \frac{\sqrt{\kappa_1\kappa_2}}{2}a_{1,h_w} - \sqrt{\kappa_2}b_{in} \quad (8)$$

with the Hamming weight $h_w = \{0,1,2,3\}$. The output field, and correspondingly the reflection coefficient, depend in general on h_w. Using the input-output theory in [6], we get reflection coefficient $r(\omega; h_\omega)$

$$r(\omega;h_\omega) = 1 - 2\left(\Delta_{d1}\kappa_2 + \Delta_{d2}\kappa_1 + (3-2h_w)\left(\kappa_1\chi_1 + \kappa_2\chi_2 - 2\sqrt{\kappa_1\kappa_2}\chi_{12}\right)\right) \times \left(\Delta_{d1}\kappa_2 + \Delta_{d2}\kappa_1 + (3-2h_w)\left(\kappa_1\chi_1 + \kappa_2\chi_2 - 2\sqrt{\kappa_1\kappa_2}\chi_{12}\right) + 2i[\Delta_{d1}\Delta_{d2} + (3-2h_w)(\Delta_{d1}\chi_1 + \Delta_{d2}\chi_2) + (3-2h_w)^2(\chi_1\chi_2 - {\chi_{12}}^2)]\right)^{-1} \quad (9)$$

with detunings $\Delta_{di} = \omega - \omega_i, i = 1,2$. In order to realize the parity measurement, we should satisfy not only the condition that the reflection coefficient depends only on the Hamming weight, but also that it has different values between even and odd parity; $r(\omega; h_\omega = 0) = r(\omega; h_\omega = 2) = r_{even}$, $r(\omega; h_\omega = 1) = r(\omega; h_\omega = 3) = r_{odd}$ and $r_{even} \neq r_{odd}$. To do so, the detunings have to satisfy the following relations;

$$\Delta_{d1} = \omega - \omega_1 = \pm\sqrt{3}\sqrt{\frac{\kappa_1}{\kappa_2}}\sqrt{\chi_1\chi_2 - \chi_{12}{}^2} \tag{10}$$

$$\Delta_{d2} = \omega - \omega_2 = \mp\sqrt{3}\sqrt{\frac{\kappa_1}{\kappa_2}}\sqrt{\chi_1\chi_2 - \chi_{12}{}^2} \tag{11}$$

Here, I point out that $\chi_1\chi_2$ is always smaller than χ_{12}^2 for transmons.
From these conditions, we would say that the reflection coefficient depends on the Hamming weight by properly choosing the resonators' frequencies [6].

2.2b Parity Measurement Condition for the System with TCQs

In our scheme, a TCQ is coupled to two bosonic modes, namely, we can consider the Hamiltonian of the system as

$$H = H_{TCQeff} + \sum_{i=1}^{2} \omega_{\pm} a_{\pm}^{\dagger} a_{\pm} + \sum_{i=1}^{2}\sum_{\pm} g_{i\pm}\left(a_{\pm} b_{\pm}^{\dagger} + \mathrm{H.c.}\right) \tag{12}$$

with the bare coupling coefficients $g_{i\pm}$ and H.c. stands for Hermitian conjugate. In order to obtain a form similar to Eq. 6, we will follow a similar procedure as we did in the subsection 2.1. I avoid the full derivation here, but the detailed derivation is written in [6]. In the end, we obtain the Hamiltonian

$$H = \frac{\widetilde{\Omega}_-}{2}\sigma^z + \sum_{i=1}^{2}(\overline{\omega_i} + \chi_i\sigma^z)\, a_i^{\dagger}a_i + [\bar{\chi}_{12} + \chi_{12}\sigma_i^z]\left(a_1 a_2^{\dagger} + H.c.\right), \tag{13}$$

where $\overline{\omega_i} = \omega_i + \frac{\chi_{i,0_+1_-} - \chi_{i,0_+0_-}}{2}$, $\chi_i = \frac{\chi_{i,0_+1_-} + \chi_{i,0_+0_-}}{2}$, $\bar{\chi}_{12} = \frac{\chi_{12,0_+1_-} - \chi_{12,0_+0_-}}{2}$ and $\chi_{12} = \frac{\chi_{12,0_+1_-} + \chi_{12,0_+0_-}}{2}$ $i = 1,2$, with $\chi_{i,0_+1_-} = \frac{\tilde{g}_{i-}{}^2}{\tilde{\Delta}_{i-}} - \frac{\left(\sqrt{2}\tilde{g}_{i-}\right)^2}{\tilde{\Delta}_{i-} + \tilde{\delta}_-} - \frac{\tilde{g}_{i+}{}^2}{\tilde{\Delta}_{i+} + \tilde{\delta}_-}$, $\chi_{i,0_+0_-} = \frac{\tilde{g}_{i+}{}^2}{\tilde{\Delta}_{i+}} + \frac{\tilde{g}_{i-}{}^2}{\tilde{\Delta}_{i-}}$, $\chi_{12,0_+1_-} = \frac{\tilde{g}_{1-}\tilde{g}_{2-}}{2}\left(\frac{1}{\tilde{\Delta}_{1-}} + \frac{1}{\tilde{\Delta}_{2-}}\right) - \frac{(\sqrt{2}\tilde{g}_{1-})(\sqrt{2}\tilde{g}_{2-})}{2}\left(\frac{1}{\tilde{\Delta}_{1-} + \tilde{\delta}_-} + \frac{1}{\tilde{\Delta}_{2-} + \tilde{\delta}_-}\right) -$

$\frac{\tilde{g}_{1+}\tilde{g}_{2+}}{2}\left(\frac{1}{\tilde{\Delta}_{1+}+\delta_c}+\frac{1}{\tilde{\Delta}_{2+}+\delta_c}\right)$ and $\chi_{12,0_+0_-}=\frac{\tilde{g}_{1+}\tilde{g}_{2+}}{2}\left(\frac{1}{\tilde{\Delta}_{1+}}+\frac{1}{\tilde{\Delta}_{2+}}\right)+\frac{\tilde{g}_{1-}\tilde{g}_{2-}}{2}\left(\frac{1}{\tilde{\Delta}_{1-}}+\frac{1}{\tilde{\Delta}_{2-}}\right)$ $i=1,2$. In addition, here, we defined $\tilde{\Delta}_{i\pm}=\tilde{\omega}_{\pm}-\omega_i$ and $\tilde{g}_{i\pm}=g_{i+}\cos(\lambda)\mp g_{i-}\sin(\lambda)$. As we can see, we can select the parameters such that the quantum switch becomes zero with the TCQ. This condition is met by setting $\tilde{g}_{1-}=0$ and $\tilde{g}_{2+}=0$ (or symmetrically, $\tilde{g}_{1+}=0$ and $\tilde{g}_{2-}=0$) [6].

As for a possible implementation, we can select $\lambda=\pi/4$, in which case the coupling coefficient between the bare transmons is much larger than the detunings [6]. In this case, the coupling coefficients should satisfy the following conditions;

$$g_{1+}=-g_{1-}=g_1, \tag{14a}$$
$$g_{1+}=g_{2-}=g_2, \tag{14b}$$

by which we get the effective coupling coefficients $\tilde{g}_{1-}=\tilde{g}_{2+}=0$, $\tilde{g}_{1+}=\sqrt{2}g_1$ and $\tilde{g}_{2-}=\sqrt{2}g_2$.Thus, the coupling coefficients need to have the sign flip, which results in zero χ_{12}. Later, these conditions are used to derive the multiport impedance.

3. Multiport Impedance

3.1 Impedance Representation of a Linear Lossless Reciprocal Multiport

In our model, we assume that only lossless components are relevant and thus we consider the equivalent Foster circuit, which is linear, lossless and reciprocal, as depicted in Figure. 2. The impedance of this canonical circuit is derived in [8] and reads

$$\boldsymbol{Z}_F(\omega)=\frac{1}{j\omega C_0}\boldsymbol{B}_0+\sum_{v=1}^{N}\frac{1}{j\omega C_v}\frac{\omega^2}{\omega^2-{\omega_v}^2}\boldsymbol{B}_v \tag{15}$$

with

$$\boldsymbol{B}_v=\begin{bmatrix} {n_{v1}}^2 & n_{v1}n_{v2} & \cdots & n_{v1}n_{vM} \\ n_{v2}n_{v1} & {n_{v2}}^2 & & n_{v2}n_{vM} \\ \vdots & & \ddots & \vdots \\ n_{vM}n_{v1} & n_{vM}n_{v2} & \cdots & {n_{vM}}^2 \end{bmatrix}$$

where the n_{vi} are the turns ratios of ideal transformers.
By exploiting this description and cleverly selecting turns ratios, we can obtain the impedance of our system, which realizes the parity measurement.

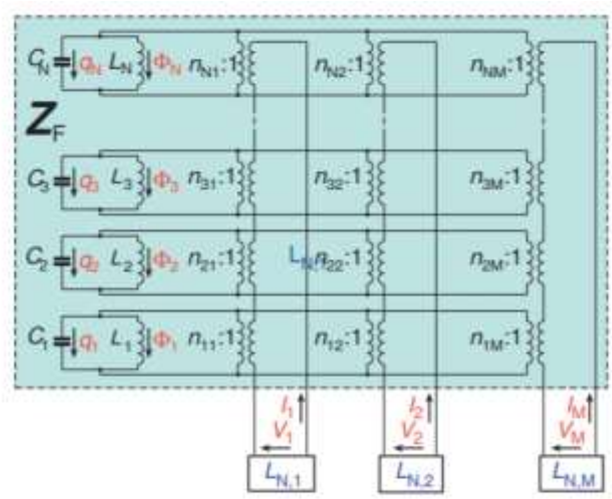

Figure. 2 Foster impedance representation of the multiport (used from [8])

3.2 TCQ Coupled to Two Resonators Model

In our model, only two modes are taken into account, so that we build the circuit model without a drive port (6-port model) and one with a drive port (7-port model), which can be intuitively understood, as shown in Figure. 4 and Figure. 5, respectively (on page 17). In these models, three TCQs are coupled, via the coupling capacitances C_{ci}, to two bosonic modes, which are depicted on the left. As for the 7-port model, we put an additional port on the right coupled to two bosonic modes via the capacitance C_t. For simplicity, I assume that the capacitances in each TCQ are symmetric as explained in subsection 2.1. In addition, the coupling capacitances C_{ci} coupled to each TCQ are also the same. In the next sections, we drive the multiport impedance of these models.

3.3 6-port Impedance of the Circuit (without a Drive Port)

3.3a Parity Measurement Conditions in the model without a Drive Port

The model of the circuit without a drive port is shown in Figure. 4. To derive the conditions of turns ratios that satisfy the sign flip discussed in 2.2b, we derive the

Hamiltonian of the system, which is calculated in detail in Appendix A.
The calculation in Appendix gives the Hamiltonian

$$
\begin{aligned}
H_{6ports} = \sum_{i=1}^{3} & \frac{C_{Ji}+C_{Ii}+C_{ci}}{2\Delta_i} {q_{J+}}^2 + \frac{C_{Ji}+C_{Ii}+C_{ci}}{2\Delta_i} {q_{J-}}^2 + \frac{C_{Ii}}{\Delta_i}\left(q_{J+}-q_{J-}\right)^2 \\
& - \frac{C_{ci}}{C_1 {\Delta_i}^2}\left(\alpha_{2i-1,1}A_i + \alpha_{2i,1}B_i\right) q_{Ji+} q_{r1} \\
& - \frac{C_{ci}}{C_2 {\Delta_i}^2}\left(\alpha_{2i-1,2}A_i + \alpha_{2i,2}B_i\right) q_{Ji+} q_{r2} \\
& - \frac{C_{ci}}{C_1 {\Delta_i}^2}\left(\alpha_{2i-1,1}B_i + \alpha_{2i,1}A_i\right) q_{Ji-} q_{r1} \\
& - \frac{C_{ci}}{C_2 {\Delta_i}^2}\left(\alpha_{2i-1,2}B_i + \alpha_{2i,2}A_i\right) q_{Ji-} q_{r2} \\
& + \frac{1}{2C_1}{q_{r1}}^2 + \frac{1}{2C_2}{q_{r2}}^2 + \frac{1}{2L_1}{\Phi_{r1}}^2 + \frac{1}{2L_2}{\Phi_{r2}}^2 \\
& - \frac{1}{C_1 C_2}\sum_{i=1}^{3} C_{ci}\left(\alpha_{2i-1,1}\alpha_{2i-1,2} + \alpha_{2i,1}\alpha_{2i,2}\right) q_{r1} q_{r2} \\
& - \sum_{i=1}^{3} E_{Ji+}\cos\left(\frac{2\pi\Phi_{Ji+}}{\Phi_0}\right) + E_{Ji-}\cos\left(\frac{2\pi\Phi_{Ji-}}{\Phi_0}\right) \qquad \text{(A8)}
\end{aligned}
$$

where $\Delta_i = \left(C_{Ji}+C_{Ii}+C_{ci}\right)^2 - {C_{Ii}}^2$, $A_i = C_{Ji}\left(C_{Ji}+2C_{Ii}\right)\left(C_{Ji}+C_{Ii}+C_{ci}\right)$ and $B_i = C_{Ji}C_{Ii}\left(C_{Ji}+2C_{Ii}\right)$. Furthermore, I calculate the coupling coefficients $g_{i\pm}^{k}$, where superscript k implies the TCQ, by introducing the annihilation and creation operators of both the harmonic oscillator modes and the transmons as Duffing oscillators. Then we obtain

$$
g_{1+}^{k} = 2e\frac{C_{ck}}{C_1 {\Delta_k}^2}\left(\alpha_{2k-1,1}A_k + \alpha_{2k,1}B_k\right)\sqrt{\frac{\hbar C_1\omega_1}{2}}\left(\frac{E_{Jk+}}{32E_{Ck}}\right)^{\frac{1}{4}} \qquad \text{(16a)}
$$

$$
g_{1-}^{k} = 2e\frac{C_{ck}}{C_1 {\Delta_k}^2}\left(\alpha_{2k-1,1}B_k + \alpha_{2k,1}A_k\right)\sqrt{\frac{\hbar C_2\omega_2}{2}}\left(\frac{E_{Jk-}}{32E_{Ck}}\right)^{\frac{1}{4}} \qquad \text{(16b)}
$$

$$g_{2+}^{k} = 2e\frac{C_{ck}}{C_2{\Delta_k}^2}\left(\alpha_{2k-1,2}A_k + \alpha_{2k,2}B_k\right)\sqrt{\frac{\hbar C_1\omega_1}{2}}\left(\frac{E_{Jk+}}{32E_{Ck}}\right)^{\frac{1}{4}} \tag{16c}$$

$$g_{2-}^{k} = 2e\frac{C_{ck}}{C_2{\Delta_k}^2}\left(\alpha_{2k-1,2}B_k + \alpha_{2k,2}A_k\right)\sqrt{\frac{\hbar C_2\omega_2}{2}}\left(\frac{E_{Jk-}}{32E_{Ck}}\right)^{\frac{1}{4}} \tag{16d}$$

In order to achieve zero χ_{12}, from subsection 2.2b, the Eq. 14 are needed. On the other hand, we can also delete χ_{12} by the following settings;

$$g_{1+}^{k} = -g_{1-}^{k} \tag{17a}$$
$$g_{2+}^{k} = g_{2-}^{k} \tag{17b}$$

Thus, using these relations, we finally obtain the conditions.

$$\alpha_{2k-1,1} = -\alpha_{2k,1}, \tag{18a}$$
$$\alpha_{2k-1,2} = \alpha_{2k,2} \tag{18b}$$

Here, we point out that these conditions make mode-mode coupling term, i.e. $-\frac{1}{C_1C_2}\sum_{i=1}^{3} C_{ci}\left(\alpha_{2i-1,1}\alpha_{2i-1,2} + \alpha_{2i,1}\alpha_{2i,2}\right)$ zero. Therefore, we would say that the two resonators do not interact with each other in this model.

3.3b Estimate for Parity Measurement

From the conditions we derived in the previous subsection, we can choose $\alpha_{2k-1,1} = -\alpha_{2k,1} = 1$ and $\alpha_{2k-1,2} = \alpha_{2k,2} = 1$. The coupling coefficients then become

$$g_{1+}^{k} = 2e\frac{C_{ck}}{C_1{\Delta_k}^2}(A_k - B_k)\sqrt{\frac{\hbar C_1\omega_1}{2}}\left(\frac{E_{Jk+}}{32E_{Ck}}\right)^{\frac{1}{4}} \tag{19a}$$

$$g_{1-}^{k} = -2e\frac{C_{ck}}{C_1{\Delta_k}^2}(A_k - B_k)\sqrt{\frac{\hbar C_2\omega_2}{2}}\left(\frac{E_{Jk-}}{32E_{Ck}}\right)^{\frac{1}{4}} \tag{19b}$$

$$g_{2+}^{k} = 2e\frac{C_{ck}}{C_2{\Delta_k}^2}(A_k + B_k)\sqrt{\frac{\hbar C_1\omega_1}{2}}\left(\frac{E_{Jk+}}{32E_{Ck}}\right)^{\frac{1}{4}} \tag{19c}$$

$$g_{2-}^{k} = 2e\frac{C_{ck}}{C_2{\Delta_k}^2}(A_k + B_k)\sqrt{\frac{\hbar C_2\omega_2}{2}}\left(\frac{E_{Jk-}}{32E_{Ck}}\right)^{\frac{1}{4}}. \tag{19d}$$

Thus, consequently, the dispersive shifts are calculated

$$\chi_1 = \frac{{\tilde{g}_{i+}}^2}{2}\frac{\tilde{\delta}_c}{\tilde{\Delta}_{1+}(\tilde{\Delta}_{1+} + \tilde{\delta}_c)} \tag{20a}$$

$$\chi_2 = {\tilde{g}_{2-}}^2\frac{\tilde{\delta}_-}{\tilde{\Delta}_{2-}(\tilde{\Delta}_{2-} + \tilde{\delta}_-)} \tag{20b}$$

where $\tilde{g}_{1+} = \sqrt{2}g_{1+}^{k}$ and $\tilde{g}_{2-} = \sqrt{2}g_{2-}^{k}$. Note that we consider the case $\lambda = \pi/4$ here.

Now, we give an example of these terms qualitatively. In these conditions with the same decay rate $\kappa_1 = \kappa_2$, we choose the energy level between the ground state and the first excited state as $\tilde{\omega}_{\pm}/2\pi = 6000\mathrm{MHz}$, the coupling parameter $J/2\pi = -400$ MHz, the anharmonicity $\delta/2\pi = -E_C/2\pi = -300$ MHz and the resonator frequency $\omega_1/2\pi = 7500$ MHz. In addition, selecting the capacitances' value $C_J = 50$ fF, $C_c = 10$ fF, $C_I = 40$ fF and $C_1 = 424$ fF, we obtain the coupling coefficients $\tilde{g}_{1+} = 86$ MHz and $\tilde{g}_{2-} = 185$ MHz and therefore dispersive shifts $\chi_1 = 1.46$ MHz and $\chi_2 = 4.88$ MHz. At this point, I firstly calculate the coupling coefficients and dispersive shift assuming that $\omega_1 = \omega_2$. Afterwards, I update the resonator frequency $\omega_2/2\pi = \omega_1/2\pi + 2\sqrt{3\chi_1\chi_2}$ and recalculate these values to check the validity.

Furthermore, we confirm whether the parity measurement can be done with this model. In this calculation, I select the following parameters listed in the Table. 1. Here I set the capacitances such that we obtain the same dispersive shifts χ_1 and χ_2, for all TCQs. As for the decay rate κ_1 and κ_2, I choose $\kappa_1 = \kappa_2 = -2\sqrt{\chi_1\chi_2}$, which gives optimal information in terms of information gain [6]. In Figure. 3, we clearly see that $r(\omega; h_\omega = 0) = r(\omega; h_\omega = 2) = r_{even}$, $r(\omega; h_\omega = 1) = r(\omega; h_\omega = 3) = r_{odd}$ and $r_{even} \neq r_{odd}$, when $\omega = \omega_1 + \sqrt{3}\sqrt{\chi_1\chi_2} = \omega_2 - \sqrt{3}\sqrt{\chi_1\chi_2} = 2\pi(75048\ \mathrm{MHz})$.

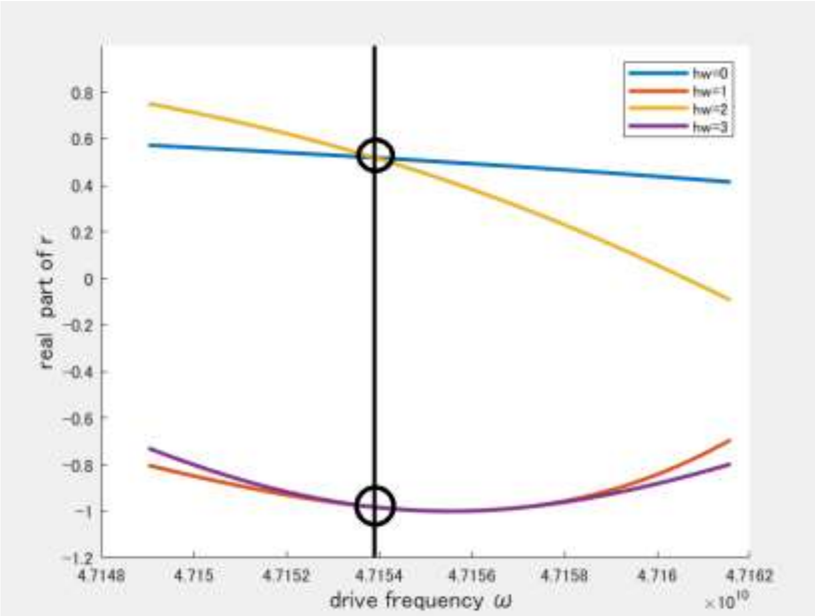

Figure. 3(a) Real part of the reflection coefficient depending on the Hamming weight

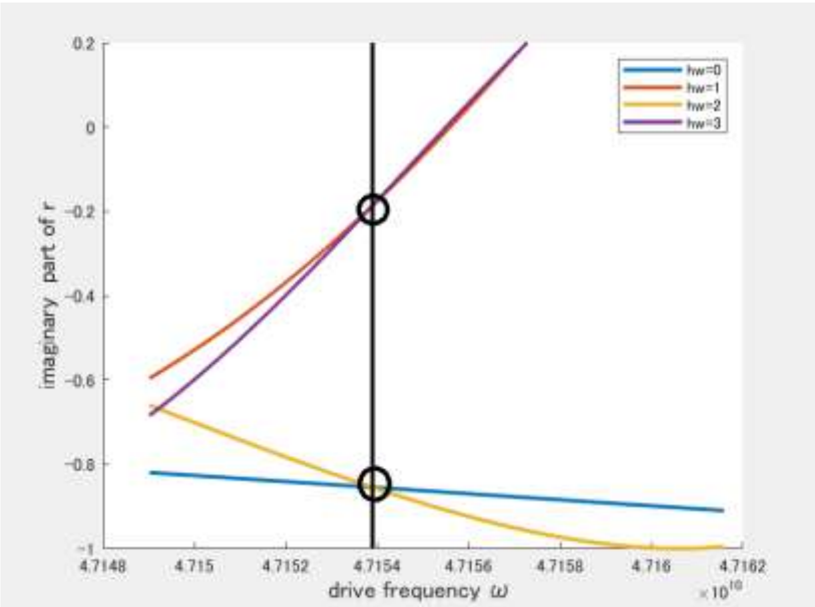

Figure. 3(b) Imaginary part of the reflection coefficient depending on the Hamming weight

Figure. 3 The difference of reflection coefficients depending on the parity.

Table.1 The list of parameters to check the parity measurement

	$\omega_{TCQ}/2\pi$ [MHz]	C_J [fF]	Cc [fF]	C_I [fF]	$\chi_1/2\pi$ [MHz]	$\chi_2/2\pi$ [MHz]
TCQa	6000	50.0	10.2	40.0	-1.5	-5.0
TCQb	5800	50.8	12.4	38.5	-1.5	-5.0
TCQc	5600	51.0	14.8	37.0	-1.5	-5.0

$J/2\pi$ [MHz]	-400	C_1 [fF]	424.4	$\kappa_1/2\pi$ [MHz]	5.5
$\delta_c/2\pi$ [MHz]	-300	C_2 [fF]	423.8	$\kappa_2/2\pi$ [MHz]	5.5
$\omega_1/2\pi$ [MHz]	7500				
$\omega_2/2\pi$ [MHz]	7510				

3.3c 6-port Impedance

In the previous subsection, we understood that the parity measurement condition can be fulfilled by the model. Thus, we take the next step to derive the impedance of this model. Following the formula Eq.15, we can derive the 6-port impedance

$$Z_{6ports}(\omega) = \sum_{v=1}^{2} \frac{1}{j\omega C_v} \frac{\omega^2}{\omega^2 - \omega_v{}^2} \boldsymbol{B}_v \tag{21}$$

with $\boldsymbol{B}_1 = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 \end{bmatrix}$ and $\boldsymbol{B}_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$.

3.4 7-port Impedance of the circuit (with a Drive Port)

3.4a Parity Measurement Conditions in the model with a Drive Port

The model of the circuit with a drive port is shown in Figure. 5. Here, a similar procedure of 3.3a is done to derive the Hamiltonian of the system. In Appendix B, the detailed calculation is given. Then, the Hamiltonian of the system reads

$$H_{7ports} = H_{6ports} + \left[-\frac{\beta_1}{C_1} + \frac{1}{C_1 C_2}\sum_{i=1}^{3} C_{ci}\left(\alpha_{2i-1,1}\alpha_{2i-1,2} + \alpha_{2i,1}\alpha_{2i,2}\right)\right] q_{r1}q_d$$

$$+ \left[-\frac{\beta_2}{C_2} + \frac{1}{C_1 C_2}\sum_{i=1}^{3} C_{ci}\left(\alpha_{2i-1,1}\alpha_{2i-1,2} + \alpha_{2i,1}\alpha_{2i,2}\right)\right] q_{r2}q_d$$

$$+ \left[\frac{1}{C_t} - \frac{\beta_1^2}{2C_1^2}(1 - C_t\beta_1^2) - \frac{\beta_2^2}{2C_2^2}(1 - C_t\beta_2^2)\right.$$

$$\left. + \frac{\beta_1\beta_2}{C_1 C_2}\sum_{i=1}^{3} C_{ci}\left(\alpha_{2i-1,1}\alpha_{2i-1,2} + \alpha_{2i,1}\alpha_{2i,2}\right)\right] q_d^2$$

$$+ \sum_{i=1}^{3} - \left[\frac{\beta_1 C_{ci}}{C_1 {\Delta_i}^2}\left(\alpha_{2i-1,1}A_i + \alpha_{2i,1}B_i\right)\right.$$

$$\left. + \frac{\beta_2 C_{ci}}{C_2 {\Delta_i}^2}\left(\alpha_{2i-1,2}A_i + \alpha_{2i,2}B_i\right)\right] q_{Ji+}q_d$$

$$- \left[\frac{\beta_1 C_{ci}}{C_1 {\Delta_i}^2}\left(\alpha_{2i-1,1}B_i + \alpha_{2i,1}A_i\right)\right.$$

$$\left. + \frac{\beta_2 C_{ci}}{C_2 {\Delta_i}^2}\left(\alpha_{2i-1,2}B_i + \alpha_{2i,2}A_i\right)\right] q_{Ji-}q_d \qquad \text{(b24)}$$

Additionally, we require $\chi_{12} = 0$, that is, $\alpha_{2k-1,1} = -\alpha_{2k,1}$ and $\alpha_{2k-1,2} = \alpha_{2k,2}$, we can rewrite the Hamiltonian as

$$H_{7ports} = H_{6ports} - \frac{\beta_1}{C_1} q_{r1}q_d - \frac{\beta_2}{C_2} q_{r2}q_d$$

$$+ \left[\frac{1}{C_t} - \frac{\beta_1^2}{2C_1^2}(1 - C_t\beta_1^2) - \frac{\beta_2^2}{2C_2^2}(1 - C_t\beta_2^2)\right] q_d^2$$

$$+ \sum_{i=1}^{3} - \left[\frac{\beta_1 C_{ci}}{C_1 {\Delta_i}^2}(A_i - B_i) + \frac{\beta_2 C_{ci}}{C_2 {\Delta_i}^2}(A_i + B_i)\right] q_{Ji+}q_d$$

$$- \left[-\frac{\beta_1 C_{ci}}{C_1 {\Delta_i}^2}(A_i - B_i) + \frac{\beta_2 C_{ci}}{C_2 {\Delta_i}^2}(A_i + B_i)\right] q_{Ji-}q_d \qquad (22)$$

Here, I note that the existence of a drive port does not change the Hamiltonian of the configuration in 3.3a. In addition, this result suggests that the driving can modify TCQs as well as modes, which makes sense, considering the fact that we can perform single-qubits gates even if the drive is coupled physically only to the resonators. Furthermore, by setting the turns ratios of 7th port β_1 and β_2, we will have the effect of drive on TCQs similar to resonators' modes.

3.4b 7-port Impedance

As we did in the section 3.3c, we calculate the 7-port impedance. For simplicity, we set $\beta_1 = \beta_2 = 1$, and the impedance of the system reads

$$Z_{7ports}(\omega) = \sum_{v=1}^{2} \frac{1}{j\omega C_v} \frac{\omega^2}{\omega^2 - \omega_v{}^2} \boldsymbol{B}_v \tag{23}$$

with $\boldsymbol{B}_1 = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{bmatrix}$ and $\boldsymbol{B}_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$.

3.5 Expansion of 7-port Impedance in the Wide Range

In this section, we expand the range of the impedance, which we calculated in the former sections. The element of the multiport impedance is defined as

$$Z_{ij} = \left.\frac{V_i}{I_j}\right|_{I_k=0 \text{ for } k\neq j} . \tag{24}$$

That is to say, Z_{ij} can be obtained by driving port j with the current I_j, while all other ports are open, i.e. $I_k = 0$ for k ≠ j [11].

Now, from this definition, we can extend the range of the impedance including the coupling capacitances C_c and C_t, depicted in Figure. 6. Consequently, the 7-port impedance reads

$$\boldsymbol{Z'}_{7ports}(\omega) = -\frac{1}{j\omega}\boldsymbol{B}_0 + \sum_{\nu=1}^{2} \frac{1}{j\omega C_\nu} \frac{\omega^2}{\omega^2 - {\omega_\nu}^2} \boldsymbol{B}_\nu \tag{25}$$

with $\boldsymbol{B}_0 = \begin{bmatrix} 1/\mathrm{C_{c1}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/\mathrm{C_{c1}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/\mathrm{C_{c2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/\mathrm{C_{c2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/\mathrm{C_{c3}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/\mathrm{C_{c3}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/\mathrm{C_t} \end{bmatrix}$,

$\boldsymbol{B}_1 = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{bmatrix}$ and $\boldsymbol{B}_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$.

4 Conclusion

In this paper, we derived the multiport impedance of the specific parity measurement network, where TCQs are used to delete unwanted quantum switch terms and coupled to two resonators. By imposing on our circuit models the parity conditions proposed in [6], the parity measurement can be done successfully. Therefore, the impedance derived from the circuit models will also provide the response of the system.

In addition, in the process of the calculation, we see that the two resonators do not interact at all with each other under the sign flip condition. We also obtain the outcome that the existence of the drive port does not change the Hamiltonian of the circuit without a drive port, while we still can have interaction of driving with TCQs and modes.

To sum up, I hope this work will help to acquire more understanding of the way to make better superconducting qubits and eventually to realize the fault tolerant quantum computing.

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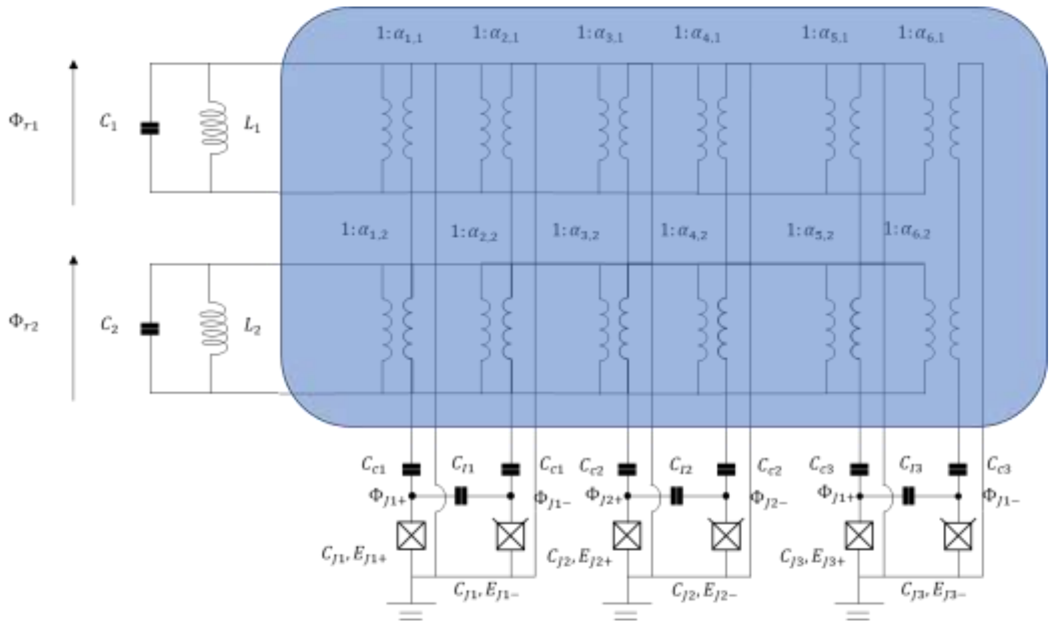

Figure. 4 Circuit model without a drive port

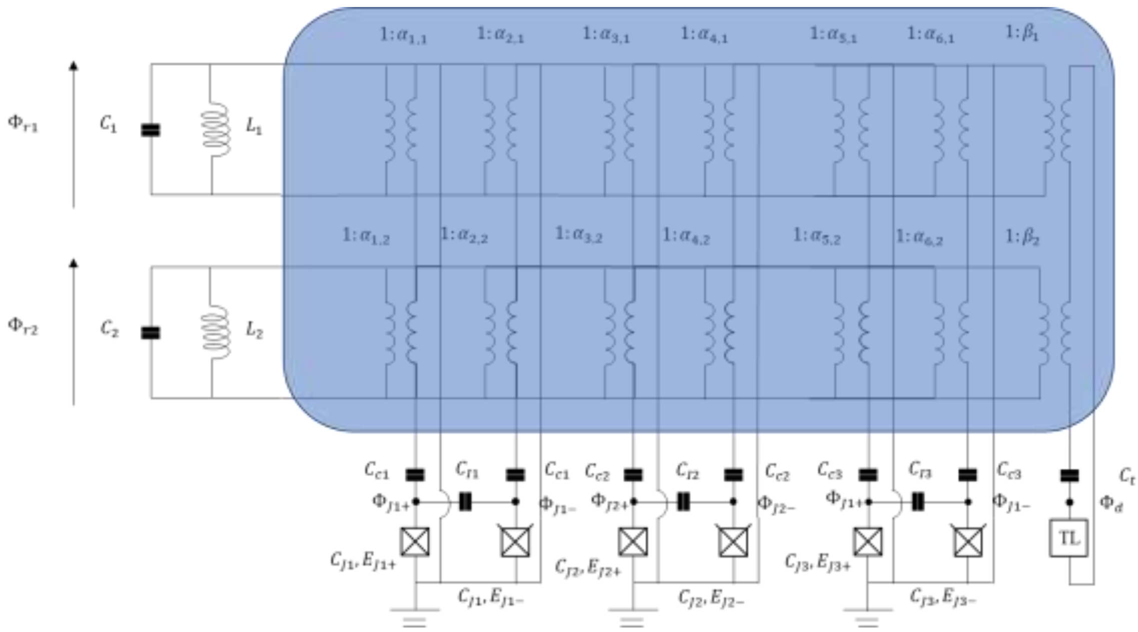

Figure. 5 Circuit model with a drive port. The TL in the lower right stands for Transmission line.

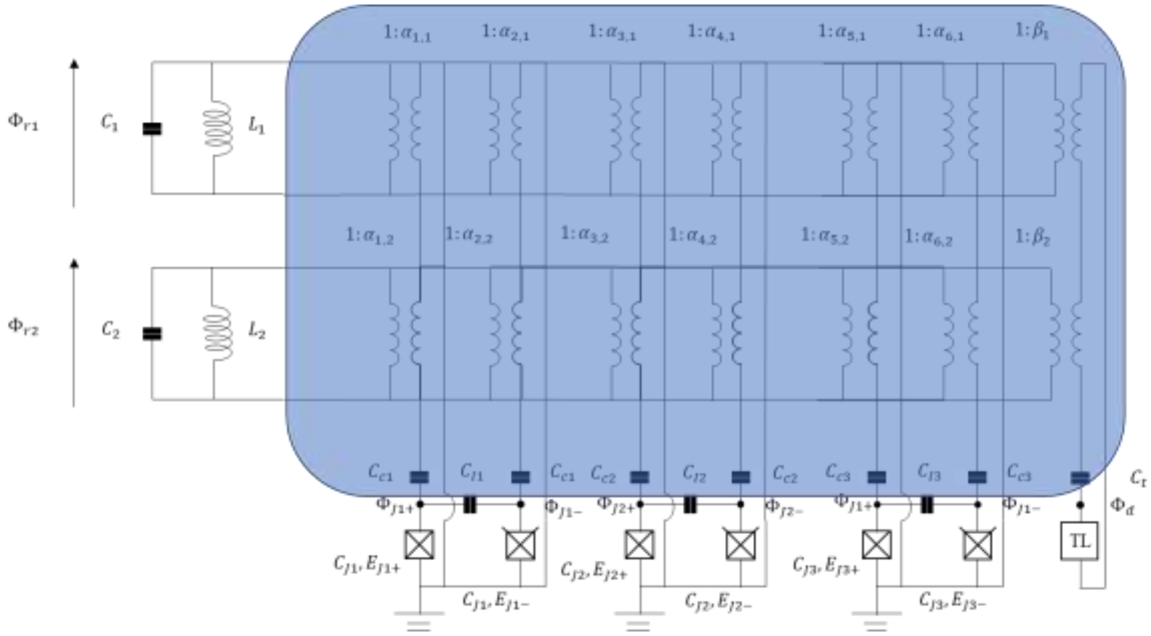

Figure. 6 Circuit model with a drive port in a wider range.

Appendix

Appendix A the Hamiltonian of the Circuit without a Drive Port

In this appendix, we calculate the Hamiltonian of the model of the circuit without a drive port, from which we can determine the conditions to obtain zero χ_{12}. The model circuit is shown in Figure. 4. In this configuration, the Lagrangian can be written as

$$\mathcal{L}_{6ports} = \sum_{i=1}^{3} \frac{C_{Ji}}{2}\dot{\Phi}_{Ji+}{}^{2} + \frac{C_{Ji}}{2}\dot{\Phi}_{Ji-}{}^{2} + \frac{C_{Ii}}{2}\left(\dot{\Phi}_{Ji+} - \dot{\Phi}_{Ji-}\right)^{2}$$
$$+ \frac{C_{ci}}{2}\left(\dot{\Phi}_{Ji+} + \alpha_{2i-1,1}\dot{\Phi}_{r1} + \alpha_{2i-1,2}\dot{\Phi}_{r2}\right)^{2}$$
$$+ \frac{C_{ci}}{2}\left(\dot{\Phi}_{Ji-} + \alpha_{2i,1}\dot{\Phi}_{r1} + \alpha_{2i,2}\dot{\Phi}_{r2}\right)^{2} + \sum_{k=1}^{2} \frac{C_k}{2}\dot{\Phi}_{k}{}^{2} - \frac{1}{2L_k}\Phi_{rk}{}^{2}$$
$$+ \sum_{i=1}^{3} E_{Ji+}\cos\left(\frac{2\pi\Phi_{Ji+}}{\Phi_0}\right) + E_{Ji+}\cos\left(\frac{2\pi\Phi_{Ji-}}{\Phi_0}\right) \qquad \text{(A1)}$$

where Φ_0 is flux quantum given by $\Phi_0 = \frac{h}{2e_0}$. In addition, the subscript of this Hamiltonian comes from the fact that we will derive a 6 by 6 matrix in the end.

The canonical momenta are given by

$$q_{Ji+} \equiv \frac{\delta\mathcal{L}}{\delta\dot{\Phi}_{Ji+}} = C_{Ji}\dot{\Phi}_{J+} + C_{Ii}\left(\dot{\Phi}_{Ji+} - \dot{\Phi}_{Ji-}\right) + C_{ci}\left(\dot{\Phi}_{Ji+} + \alpha_{2i-1,1}\dot{\Phi}_{r1} + \alpha_{2i-1,2}\dot{\Phi}_{r2}\right), \text{(A2a)}$$

$$q_{Ji-} \equiv \frac{\delta\mathcal{L}}{\delta\dot{\Phi}_{Ji-}} = \mathrm{C_{Ji}}\dot{\Phi}_{Ji-} - C_{Ii}\left(\dot{\Phi}_{Ji+} - \dot{\Phi}_{Ji-}\right) + C_{ci}\left(\dot{\Phi}_{Ji-} + \alpha_{2i,1}\dot{\Phi}_{r1} + \alpha_{2i,2}\dot{\Phi}_{r2}\right), \text{(A2b)}$$

$$q_{r1} \equiv \frac{\delta\mathcal{L}}{\delta\dot{\Phi}_{r1}} = \mathrm{C_1}\dot{\Phi}_{r1} + \sum_{i=1}^{3} C_{ci}\,\alpha_{2i-1,1}\left(\dot{\Phi}_{Ji+} + \alpha_{2i-1,1}\dot{\Phi}_{r1} + \alpha_{2i-1,2}\dot{\Phi}_{r2}\right)$$
$$+ C_{ci}\alpha_{2i,1}\left(\dot{\Phi}_{Ji-} + \alpha_{2i,1}\dot{\Phi}_{r1} + \alpha_{2i,2}\dot{\Phi}_{r2}\right), \qquad \text{(A2c)}$$

$$q_{r2} \equiv \frac{\delta \mathcal{L}}{\delta \dot{\Phi}_{r2}} = \mathrm{C}_2 \dot{\Phi}_{r2} + \sum_{i=1}^{3} C_{ci}\, \alpha_{2i-1,2}\big(\dot{\Phi}_{Ji+} + \alpha_{2i-1,1}\dot{\Phi}_{r1} + \alpha_{2i-1,2}\dot{\Phi}_{r2}\big) + C_{ci}\alpha_{2i,2}\big(\dot{\Phi}_{Ji-} + \alpha_{2i,1}\dot{\Phi}_{r1} + \alpha_{2i,2}\dot{\Phi}_{r2}\big) \tag{A2d}$$

Thus, the Hamiltonian is then derived as the Legendre transform

$$H = \sum_{\mathrm{i}=1}^{3} q_{Ji+}\dot{\Phi}_{Ji+} + q_{Ji-}\dot{\Phi}_{Ji-} + \sum_{k=1}^{2} q_{rk}\dot{\Phi}_{rk} - \mathcal{L} \tag{A3}$$

Here I introduce the matrix form of the Hamiltonian, namely,

$$\boldsymbol{q} = \boldsymbol{C_{6ports}}\dot{\boldsymbol{\Phi}} \tag{A4}$$

with $\boldsymbol{q} = \begin{bmatrix} q_{J1+} \\ q_{J1-} \\ q_{J2+} \\ q_{J2-} \\ q_{J3+} \\ q_{J3-} \\ q_{r1} \\ q_{r2} \end{bmatrix}$ and $\dot{\boldsymbol{\Phi}} = \begin{bmatrix} \dot{\Phi}_{J1+} \\ \dot{\Phi}_{J1-} \\ \dot{\Phi}_{J2+} \\ \dot{\Phi}_{J2-} \\ \dot{\Phi}_{J3+} \\ \dot{\Phi}_{J3-} \\ \dot{\Phi}_{r1} \\ \dot{\Phi}_{r2} \end{bmatrix}$.

In addition, $\boldsymbol{C_{6ports}} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B}^T & \boldsymbol{D} \end{bmatrix}$ with $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A_1} & 0 & 0 \\ 0 & \boldsymbol{A_2} & 0 \\ 0 & 0 & \boldsymbol{A}_3 \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} \boldsymbol{B_1} \\ \boldsymbol{B_2} \\ \boldsymbol{B_3} \end{bmatrix}$ and $\boldsymbol{D} = \begin{bmatrix} C_1 + \sum_{i=1}^{3} C_{ci}(\alpha_{2i-1,1}^2 + \alpha_{2i,1}^2) & \sum_{i=1}^{3} C_{ci}(\alpha_{2i-1,1}\alpha_{2i-1,2} + \alpha_{2i,1}\alpha_{2i,2}) \\ \sum_{i=1}^{3} C_{ci}(\alpha_{2i-1,1}\alpha_{2i-1,2} + \alpha_{2i,1}\alpha_{2i,2}) & C_2 + \sum_{i=1}^{3} C_{ci}\big(\alpha_{2i-1,2}^2 + \alpha_{2i,2}^2\big) \end{bmatrix}$, where $\boldsymbol{A_i} = \begin{bmatrix} C_{Ji} + C_{Ii} + C_{ci} & -C_{Ii} \\ -C_{Ii} & C_{Ji} + C_{Ii} + C_{ci} \end{bmatrix}$ and $\boldsymbol{B_i} = \begin{bmatrix} C_{ci}\alpha_{2i-1,1} & C_{ci}\alpha_{2i-1,2} \\ C_{ci}\alpha_{2i,1} & C_{ci}\alpha_{2i,2} \end{bmatrix}$.

In order to invert the capacitance matrix $\boldsymbol{C}$, we implement Block Matrix Inversion, that is,

$$\boldsymbol{C}^{-1} = \begin{bmatrix} \boldsymbol{A^{-1}} + \boldsymbol{A^{-1}B}(\boldsymbol{D} - \boldsymbol{B^T A^{-1} B})^{-1}\boldsymbol{B}^T\boldsymbol{A^{-1}} & -\boldsymbol{A^{-1}B}(\boldsymbol{D} - \boldsymbol{B^T A^{-1} B})^{-1} \\ -(\boldsymbol{D} - \boldsymbol{B^T A^{-1} B})^{-1}\boldsymbol{B^T A^{-1}} & (\boldsymbol{D} - \boldsymbol{B^T A^{-1} B})^{-1} \end{bmatrix} \tag{A5}$$

from which we obtain the inverse of the capacitance matrix

$$\boldsymbol{C}_{\boldsymbol{6port}}^{-\boldsymbol{1}} = \begin{bmatrix} \boldsymbol{A}' & \boldsymbol{B}' \\ \boldsymbol{B}'^{\boldsymbol{T}} & \boldsymbol{D}' \end{bmatrix} \quad \text{(A6)}$$

where $\boldsymbol{A}' = \begin{bmatrix} \boldsymbol{A}'_{\boldsymbol{1}} & 0 & 0 \\ 0 & \boldsymbol{A}'_{\boldsymbol{2}} & 0 \\ 0 & 0 & \boldsymbol{A}'_3 \end{bmatrix}$ $\boldsymbol{B}' = \begin{bmatrix} \boldsymbol{B}'_{\boldsymbol{1}} \\ \boldsymbol{B}'_{\boldsymbol{2}} \\ \boldsymbol{B}'_{\boldsymbol{3}} \end{bmatrix}$ and $\boldsymbol{D} =$

$$\begin{bmatrix} \frac{1}{C_1}(1 - \frac{1}{C_1}\sum_{i=1}^{3} C_{ci}(\alpha_{2i-1,1}^2 + \alpha_{2i,1}^2)) & -\frac{1}{C_1C_2}\sum_{i=1}^{3} C_{ci}(\alpha_{2i-1,1}\alpha_{2i-1,2} + \alpha_{2i,1}\alpha_{2i,2}) \\ -\frac{1}{C_1C_2}\sum_{i=1}^{3} C_{ci}(\alpha_{2i-1,1}\alpha_{2i-1,2} + \alpha_{2i,1}\alpha_{2i,2}) & \frac{1}{C_2}(1 - \frac{1}{C_2}\sum_{i=1}^{3} C_{ci}(\alpha_{2i-1,2}^2 + \alpha_{2i,2}^2)) \end{bmatrix}$$

, with $\boldsymbol{A}'_{\boldsymbol{i}} = \begin{bmatrix} \frac{C_{Ji}+C_{Ii}+C_{ci}}{\Delta_i} & \frac{C_{Ii}}{\Delta_i} \\ \frac{C_{Ii}}{\Delta_i} & \frac{C_{Ji}+C_{Ii}+C_{ci}}{\Delta_i} \end{bmatrix}$, $\Delta_i = (C_{Ji} + C_{Ii} + C_{ci})^2 - C_{Ii}^2$ and $\boldsymbol{B}'_{\boldsymbol{i}} =$

$$\begin{bmatrix} -\frac{C_{ci}}{C_1\Delta_i}\{\alpha_{2i-1,1}(C_{Ji} + C_{Ii} + C_{ci}) + \alpha_{2i,1}C_{Ii}\} & -\frac{C_{ci}}{C_2\Delta_i}\{\alpha_{2i-1,2}(C_{Ji} + C_{Ii} + C_{ci}) + \alpha_{2i,2}C_{Ii}\} \\ -\frac{C_{ci}}{C_1\Delta_i}\{\alpha_{2i-1,1}C_{Ii} + \alpha_{2i,1}(C_{Ji} + C_{Ii} + C_{ci})\} & -\frac{C_{ci}}{C_2\Delta_i}\{\alpha_{2i-1,2}C_{Ii} + \alpha_{2i,2}(C_{Ji} + C_{Ii} + C_{ci})\} \end{bmatrix}$$

.

At this point, we make a weak coupling assumption, neglecting the terms of product of coupling capacitances over other capacitances, i.e., $C_{ck}C_{cl}/C_m$ and employing the Taylor expansion. Consequently, the canonical coordinate $\boldsymbol{\Phi}$ can be written in terms of canonical momenta $\boldsymbol{q}$ as

$$\dot{\Phi}_{Ji+} = \frac{C_{Ji} + C_{Ii} + C_{ci}}{\Delta_i} q_{Ji+} + \frac{C_{Ii}}{\Delta_i} q_{Ji-} - \frac{C_{ci}}{C_1\Delta_i}\{\alpha_{2i-1,1}(C_{Ji} + C_{Ii} + C_{ci}) + \alpha_{2i,1}C_{Ii}\}q_{r1} - \frac{C_{ci}}{C_2\Delta_i}\{\alpha_{2i-1,2}(C_{Ji} + C_{Ii} + C_{ci}) + \alpha_{2i,2}C_{Ii}\}q_{r2}, \quad \text{(A7a)}$$

$$\dot{\Phi}_{Ji-} = \frac{C_{Ii}}{\Delta_i} q_{Ji+} + \frac{C_{Ji} + C_{Ii} + C_{ci}}{\Delta_i} q_{Ji-} - \frac{C_{ci}}{C_1\Delta_i}\{\alpha_{2i-1,1}C_{Ii} + \alpha_{2i,1}(C_{Ji} + C_{Ii} + C_{ci})\}q_{r1} - \frac{C_{ci}}{C_2\Delta_i}\{\alpha_{2i-1,2}C_{Ii} + \alpha_{2i,2}(C_{Ji} + C_{Ii} + C_{ci})\}q_{r2}, \quad \text{(A7b)}$$

$$\dot{\Phi}_{r1} = \frac{1}{C_1}(1 - \frac{1}{C_1}\sum_{i=1}^{3} C_{ci}(\alpha_{2i-1,1}^2 + \alpha_{2i,1}^2))q_{r1}$$

$$- \frac{1}{C_1 C_2}\sum_{i=1}^{3} C_{ci}\left(\alpha_{2i-1,1}\alpha_{2i-1,2} + \alpha_{2i,1}\alpha_{2i,2}\right) q_{r2}$$

$$+ \sum_{i=1}^{3} - \frac{C_{ci}}{C_1\Delta_i}\{\alpha_{2i-1,1}\left(C_{Ji} + C_{Ii} + C_{ci}\right) + \alpha_{2i,1}C_{Ii}\}q_{Ji+}$$

$$- \frac{C_{ci}}{C_1\Delta_i}\{\alpha_{2i-1,1}C_{Ii} + \alpha_{2i,1}\left(C_{Ji} + C_{Ii} + C_{ci}\right)\}q_{Ji-}, \qquad \text{(A7c)}$$

$$\dot{\Phi}_{r2} = \frac{1}{C_2}(1 - \frac{1}{C_2}\sum_{i=1}^{3} C_{ci}\left(\alpha_{2i-1,2}^2 + \alpha_{2i,2}^2\right))\, q_{r2}$$

$$- \frac{1}{C_1 C_2}\sum_{i=1}^{3} C_{ci}(\alpha_{2i-1,1}\alpha_{2i-1,2} + \alpha_{2i,1}\alpha_{2i,2})q_{r1}$$

$$+ \sum_{i=1}^{3} - \frac{C_{ci}}{C_2\Delta_i}\{\alpha_{2i-1,2}\left(C_{Ji} + C_{Ii} + C_{ci}\right) + \alpha_{2i,2}C_{Ii}\}q_{Ji+}$$

$$- \frac{C_{ci}}{C_2\Delta_i}\{\alpha_{2i-1,2}C_{Ii} + \alpha_{2i,2}\left(C_{Ji} + C_{Ii} + C_{ci}\right)\}q_{Ji-} \qquad \text{(A7d)}$$

Substituting Eq. A7 and again assuming a weak coupling, we obtain the Hamiltonian

$$H_{6ports} = \sum_{i=1}^{3} \frac{C_{Ji} + C_{Ii} + C_{ci}}{2\Delta_i} {q_{J+}}^2 + \frac{C_{Ji} + C_{Ii} + C_{ci}}{2\Delta_i} {q_{J-}}^2 + \frac{C_{Ii}}{\Delta_i}\left(q_{J+} - q_{J-}\right)^2$$

$$- \frac{C_{ci}}{C_1{\Delta_i}^2}\left(\alpha_{2i-1,1}A_i + \alpha_{2i,1}B_i\right)q_{Ji+}q_{r1}$$

$$- \frac{C_{ci}}{C_2{\Delta_i}^2}\left(\alpha_{2i-1,2}A_i + \alpha_{2i,2}B_i\right)q_{Ji+}q_{r2}$$

$$- \frac{C_{ci}}{C_1{\Delta_i}^2}\left(\alpha_{2i-1,1}B_i + \alpha_{2i,1}A_i\right)q_{Ji-}q_{r1}$$

$$- \frac{C_{ci}}{C_2{\Delta_i}^2}\left(\alpha_{2i-1,2}B_i + \alpha_{2i,2}A_i\right)q_{Ji-}q_{r2}$$

$$+\frac{1}{2C_1}{q_{r1}}^2+\frac{1}{2C_2}{q_{r2}}^2+\frac{1}{2L_1}{\Phi_{r1}}^2+\frac{1}{2L_2}{\Phi_{r2}}^2$$

$$-\frac{1}{C_1C_2}\sum_{i=1}^{3}C_{ci}\left(\alpha_{2i-1,1}\alpha_{2i-1,2}+\alpha_{2i,1}\alpha_{2i,2}\right)q_{r1}q_{r2}$$

$$-\sum_{i=1}^{3}E_{Ji+}\cos\left(\frac{2\pi\Phi_{Ji+}}{\Phi_0}\right)+E_{Ji-}\cos\left(\frac{2\pi\Phi_{Ji-}}{\Phi_0}\right) \tag{A8}$$

where $\Delta_i=\left(C_{Ji}+C_{Ii}+C_{ci}\right)^2-{C_{Ii}}^2$, $A_i=C_{Ji}\left(C_{Ji}+2C_{Ii}\right)\left(C_{Ji}+C_{Ii}+C_{ci}\right)$ and $B_i=C_{Ji}C_{Ii}\left(C_{Ji}+2C_{Ii}\right)$. Here, since the resonator capacitances C_1 and C_2 are approximately 50 times larger than the coupling capacitance, I also neglect the term $C_c/C_k, k=1,2$.

Appendix B the Hamiltonian of the Circuit without a Drive Port

In this section, we calculate the Hamiltonian of the model with a drive port to extend the discussion to the 7 by 7 matrix as we did in the Appendix A. The system is illustrated in Figure. 5. Here I assume that the transmission line in the drive port does not have effect on this circuit and thus I neglect this part. The Lagrangian is written as

$$\mathcal{L}_{7ports}=\mathcal{L}_{6ports}+\mathcal{L}_{TL}+\frac{C_t}{2}\left(\dot{\Phi}_d+\beta_1\dot{\Phi}_{r1}+\beta_2\dot{\Phi}_{r2}\right)^2 \tag{B1}$$

and the canonical momenta as

$$q_d\equiv\frac{\delta\mathcal{L}}{\delta\dot{\Phi}_d}=C_t\left(\dot{\Phi}_d+\beta_1\dot{\Phi}_{r1}+\beta_2\dot{\Phi}_{r2}\right) \tag{B2a}$$

$$q_{r1}\equiv\frac{\delta\mathcal{L}}{\delta\dot{\Phi}_{r1}}=\mathrm{C}_1\dot{\Phi}_{r1}+C_t\beta_1\left(\dot{\Phi}_d+\beta_1\dot{\Phi}_{r1}+\beta_2\dot{\Phi}_{r2}\right)$$

$$+\sum_{i=1}^{3}C_{ci}\,\alpha_{2i-1,1}\left(\dot{\Phi}_{Ji+}+\alpha_{2i-1,1}\dot{\Phi}_{r1}+\alpha_{2i-1,2}\dot{\Phi}_{r2}\right)$$

$$+C_{ci}\alpha_{2i,1}\left(\dot{\Phi}_{Ji-}+\alpha_{2i,1}\dot{\Phi}_{r1}+\alpha_{2i,2}\dot{\Phi}_{r2}\right) \tag{B2b}$$

$$q_{r2} \equiv \frac{\delta \mathcal{L}}{\delta \dot{\Phi}_{r2}} = \mathrm{C}_2 \dot{\Phi}_{r2} + C_t \beta_2 \big(\dot{\Phi}_d + \beta_1 \dot{\Phi}_{r1} + \beta_2 \dot{\Phi}_{r2}\big)$$

$$+ \sum_{i=1}^{3} C_{ci}\, \alpha_{2i-1,2} \big(\dot{\Phi}_{Ji+} + \alpha_{2i-1,1} \dot{\Phi}_{r1} + \alpha_{2i-1,2} \dot{\Phi}_{r2}\big)$$

$$+ C_{ci} \alpha_{2i,2} \big(\dot{\Phi}_{Ji-} + \alpha_{2i,1} \dot{\Phi}_{r1} + \alpha_{2i,2} \dot{\Phi}_{r2}\big) \tag{B2c}$$

with Eq. A2a and A2b.

In analogy with Appendix A, I introduce the matrix form and the capacitance matrix reads

$$\boldsymbol{C_{7ports}} = \begin{bmatrix} \boldsymbol{C'_{6ports}} & \boldsymbol{E} \\ \boldsymbol{E^T} & C_t \end{bmatrix} \tag{B3}$$

where $\boldsymbol{C'_{6ports}} = \boldsymbol{C_{6ports}} + \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{F} \end{bmatrix}$ with $\boldsymbol{F} = \begin{bmatrix} C_t \beta_1^2 & C_t \beta_1 \beta_2 \\ C_t \beta_1 \beta_2 & C_t \beta_2^2 \end{bmatrix}$ and $\boldsymbol{E} = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad C_t\beta_1 \quad C_t\beta_2]^{\boldsymbol{T}}$. Following Eq. A5, assuming a weak coupling and employing the Taylor expansion, the inverse of the capacitance matrix becomes

$$\boldsymbol{C_{7ports}^{-1}} = \begin{bmatrix} \boldsymbol{C_{6ports}^{-1}} & \boldsymbol{E'} \\ \boldsymbol{E'^T} & \dfrac{1}{C_t} \end{bmatrix} \tag{B4}$$

with $\boldsymbol{E'} = [\boldsymbol{e_1} \quad \boldsymbol{e_2} \quad \boldsymbol{e_3} \quad \boldsymbol{e'}]^{\boldsymbol{T}}$, where $\boldsymbol{e_i} =$

$$\begin{bmatrix} \frac{\beta_1 C_{ci}}{C_1 \Delta_i}\{\alpha_{2i-1,1}\big(C_{Ji} + C_{Ii} + C_{ci}\big) + \alpha_{2i,1} C_{Ii}\} + \frac{\beta_2 C_{ci}}{C_2 \Delta_i}\{\alpha_{2i-1,2}\big(C_{Ji} + C_{Ii} + C_{ci}\big) + \alpha_{2i,2} C_{Ii}\} \\ \frac{\beta_1 C_{ci}}{C_1 \Delta_i}\{\alpha_{2i-1,1} C_{Ii} + \alpha_{2i,1}\big(C_{Ji} + C_{Ii} + C_{ci}\big)\} + \frac{\beta_2 C_{ci}}{C_2 \Delta_i}\{\alpha_{2i-1,2} C_{Ii} + \alpha_{2i,2}\big(C_{Ji} + C_{Ii} + C_{ci}\big)\} \end{bmatrix}^{\boldsymbol{T}}$$

and $\boldsymbol{e_i} =$

$$\begin{bmatrix} -\frac{\beta_1}{C_1}(1 - \frac{1}{C_1}\sum_{i=1}^{3} C_{ci}(\alpha_{2i-1,1}^2 + \alpha_{2i,1}^2)) + \frac{\beta_2}{C_1 C_2}\sum_{i=1}^{3} C_{ci}(\alpha_{2i-1,1}\alpha_{2i-1,2} + \alpha_{2i,1}\alpha_{2i,2}) \\ \frac{\beta_1}{C_1 C_2}\sum_{i=1}^{3} C_{ci}\big(\alpha_{2i-1,1}\alpha_{2i-1,2} + \alpha_{2i,1}\alpha_{2i,2}\big) - \frac{\beta_2}{C_2}(1 - \frac{1}{C_2}\sum_{i=1}^{3} C_{ci}\big(\alpha_{2i-1,2}^2 + \alpha_{2i,2}^2\big)) \end{bmatrix}^{\boldsymbol{T}}$$

Note that in the weak coupling assumption, I also assume C_t as a small capacitance. Then, substituting the canonical coordinates $\boldsymbol{\Phi}$ in terms of canonical momenta $\boldsymbol{q}$ into the Hamiltonian,

$$H_{7ports} = \sum_{i=1}^{3} q_{Ji+}\dot{\Phi}_{Ji+} + q_{Ji-}\dot{\Phi}_{Ji-} + \sum_{k=1}^{2} q_{rk}\dot{\Phi}_{rk} + q_d\dot{\Phi}_d - \mathcal{L}_{7ports} \tag{B5}$$

we obtain the Hamiltonian

$$\begin{aligned}
H_{7ports} = H_{6ports} &+ \left[-\frac{\beta_1}{C_1} + \frac{1}{C_1C_2}\sum_{i=1}^{3} C_{ci}\left(\alpha_{2i-1,1}\alpha_{2i-1,2} + \alpha_{2i,1}\alpha_{2i,2}\right)\right] q_{r1}q_d \\
&+ \left[-\frac{\beta_2}{C_2} + \frac{1}{C_1C_2}\sum_{i=1}^{3} C_{ci}\left(\alpha_{2i-1,1}\alpha_{2i-1,2} + \alpha_{2i,1}\alpha_{2i,2}\right)\right] q_{r2}q_d \\
&+ \left[\frac{1}{C_t} - \frac{\beta_1^2}{2C_1^2}(1 - C_t\beta_1^2) - \frac{\beta_2^2}{2C_2^2}(1 - C_t\beta_2^2)\right. \\
&\left.+ \frac{\beta_1\beta_2}{C_1C_2}\sum_{i=1}^{3} C_{ci}\left(\alpha_{2i-1,1}\alpha_{2i-1,2} + \alpha_{2i,1}\alpha_{2i,2}\right)\right] q_d^2 \\
&+ \sum_{i=1}^{3} - \left[\frac{\beta_1 C_{ci}}{C_1{\Delta_i}^2}\left(\alpha_{2i-1,1}A_i + \alpha_{2i,1}B_i\right)\right. \\
&\left.+ \frac{\beta_2 C_{ci}}{C_2{\Delta_i}^2}\left(\alpha_{2i-1,2}A_i + \alpha_{2i,2}B_i\right)\right] q_{Ji+}q_d \\
&- \left[\frac{\beta_1 C_{ci}}{C_1{\Delta_i}^2}\left(\alpha_{2i-1,1}B_i + \alpha_{2i,1}A_i\right)\right. \\
&\left.+ \frac{\beta_2 C_{ci}}{C_2{\Delta_i}^2}\left(\alpha_{2i-1,2}B_i + \alpha_{2i,2}A_i\right)\right] q_{Ji-}q_d
\end{aligned} \tag{B6}$$

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